THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES

École doctorale : MSTII - Mathématiques, Sciences et technologies de l'information, Informatique
Spécialité : Mathématiques
Unité de recherche : Institut Fourier

Limites de champ moyen en mécanique des fluides et théorie cinétique

Mean-field limits in fluid mechanics and kinetic theory

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Remerciements

Je tiens en premier lieu à remercier mes directeurs de thèse Christophe Lacave et Évelyne Miot pour leur soutien constant durant ces trois années. Leur bienveillance et leurs conseils (mathématiques, administratifs, rédactionnels...) m'ont beaucoup apporté professionnellement et personnellement.

Je remercie Daniel Han-Kwan et François Golse pour avoir accepté d'être rapporteurs de cette thèse. Je remercie également Ayman Moussa pour avoir pris part à mon comité de suivi de thèse. Merci à tous les membres du jury que je n'ai pas encore cité dans ce paragraphe, François Bolley, Mitia Duerinckx, Thierry Gallay, Mikaela Iacobelli et Laure Saint-Raymond pour avoir accepté d'y prendre part.

Merci à tous les chercheurs, doctorants ou post-doctorants avec qui j'ai eu l'occasion d'échanger lors de conférences, séminaires, pauses thé du laboratoire... Je tiens notamment à remercier Franck Sueur pour les diverses discussions que nous avons pu avoir, ainsi que Catherine Sulem pour son accueil à l'université de Toronto et son accompagnement qui m'ont permis de découvrir de jolies mathématiques.

J'adresse un immense merci aux membres des services administratifs et financiers qui m'ont permis d'envisager sereinement toutes les diverses réservations de train, fiches missions et autres tâches administratives.

Je remercie ma famille et notamment mes parents et mes sœurs pour m'avoir soutenu durant mes études ainsi que tous mes amis Grenoblois (ou non Grenoblois) grâce à qui j'ai pu vivre de très beaux moments hors de l'université.

Mon dernier paragraphe de remerciements va à tous les non-permanents du laboratoire pour toutes les pauses midi, goûters et divers évènements qui rendaient les journées plus joyeuses.

Résumé

On étudie dans cette thèse le comportement statistique de plusieurs systèmes de particules issus de la mécanique des fluides. Plus précisément, on s'intéresse au problème-type suivant : justifier la convergence d'un système modélisant l'évolution d'un nombre fini de particules vers un système modélisant l'évolution d'une densité continue lorsque le nombre de particules devient grand. On considère des modèles de "champ moyen", c'est-à-dire des modèles où les interactions entre les particules sont à "longue portée".

Au Chapitre 1 on introduit les différents modèles étudiés et les résultats principaux de la thèse, puis on présente un historique des différents travaux traitant des limites de champ moyen dans la littérature ainsi que la "méthode d'énergie modulée" introduite par Duerinckx et Serfaty dans [81] qui sera le principal outil utilisé dans les chapitres 2 et 3. Enfin, on donne davantage de précisions sur les résultats principaux de ce manuscrit et on détaille les arguments utilisés pour les démontrer.

Dans le Chapitre 2 on étudie un système d'équations aux dérivées partielles modélisant l'interaction entre un fluide incompressible non visqueux et une densité continue de solides plongés dans ce fluide et soumis à un effet gyrocinétique. On montre le caractère localement bien posé en temps des solutions fortes de ces équations puis on les obtient comme limite de champ moyen d'un modèle décrivant l'évolution d'un nombre fini de solides plongés dans un fluide. Les résultats de ce chapitre correspondent à l'article [67].

Dans le Chapitre 3 on s'intéresse aux limites de champ moyen d'un système modélisant l'évolution de petits tourbillons ou "points vortex" dans un lac de profondeur non constante. Les résultats de ce chapitre correspondent à l'article [68].

Abstract

In this thesis we study the statistical behaviour of several particle systems related to fluid dynamics. Namely we are interested in the following problem: Prove the convergence of a system modeling the evolution of a finite number of particles to a system modeling the evolution of a continuous density when the number of particles becomes very large. We consider "mean-field" models, for which particles are subjected to a long range interaction potential.

In Chapter 1 we introduce the models and the main results studied in this thesis. Then we provide an overview of the previous works that have investigated mean-field limits and we explain the main ideas of the "modulated energy method" introduced by Duerinckx and Serfaty in [81], which will be the main tool used in chapters 2 and 3. Finally we give more details about the main results of this thesis and we describe the main arguments used for the proofs of these results.

In Chapter 2 we study a system of partial differential equations modeling the interaction between an incompressible inviscid fluid and a dispersed phase of solid particles subjected to a gyroscopic force. We show the local well-posedness of strong solutions of these equations and we derivate them as the mean-field limit of a model describing the evolution of a finite number of solids in a fluid. The results of this chapter are the content of [67].

In Chapter 3 we investigate the mean-field limits of a system modeling the evolution of small vortices in a lake of non-constant topography. The results of this chapter are the content of [68].

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Chapitre 1

Introduction

Dans ce manuscrit, on s'intéresse aux limites de champ moyen de deux systèmes : un modèle décrivant un nombre finis de solides plongés dans un fluide (au Chapitre 2) et un modèle décrivant l'évolution de petits tourbillons dans un lac de profondeur variable (au Chapitre 3). Dans cette introduction on définit les différents modèles étudiés (en Section 1.1) et on expose les résultats et arguments principaux de cette thèse (en Section 1.3) ainsi que le contexte bibliographique dans lequel s'inscrivent ces résultats (en Section 1.2). Les notations principales de ce manuscrit sont rassemblées sur les pages 151 à 154.

1.1 Modèles étudiés et résultats principaux

1.1.1 Équations d'Euler incompressibles dans le plan et points vortex

Équations d'Euler incompressibles dans le plan

Les équations d'Euler incompressibles dans le plan décrivent l'évolution du champ de vitesse $u(t, x)$ d'un fluide incompressible et non visqueux où t est la variable de temps et $x \in \mathbb{R}^2$ la variable d'espace. Elles s'écrivent

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \nabla p \\ \operatorname{div}(u) = 0. \end{cases} \tag{1.1.1}$$

Dans le cas d'un domaine à bord on adjoint le plus souvent à cette équation une condition de flux nul au bord $u \cdot n = 0$. Elles ont été écrites pour la première fois par Euler dans [29]. Pour une introduction à l'obtention de ces équations on renvoie à [18, Chapitre 1], [31, Chapitre 1] ou [63, Section 1.1].

La pression p est une inconnue du problème et peut être interprétée comme un multiplicateur de Lagrange provenant de la contrainte $\operatorname{div}(u) = 0$.

En prenant la divergence de la première équation, on trouve

$$\Delta p = \mathrm{div}((u \cdot \nabla)u).$$

On peut donc exprimer la pression en fonction du champ de vitesse à condition d'avoir suffisamment d'hypothèses de régularité et de décroissance sur u pour pouvoir résoudre l'équation précédente.

Pour éviter d'avoir à traiter ce terme de pression, on peut formuler cette équation comme une équation d'évolution sur le rotationnel du champ de vitesse

$$\omega = \mathrm{curl}(u) = \partial_1 u_2 - \partial_2 u_1$$

aussi appelé "vorticité du fluide" ou "fonction tourbillon". Cette quantité peut s'interpréter comme la vitesse de rotation infinitésimale d'une particle de fluide sous l'action du champ de vitesse u. En effet, on peut localement écrire le développement de Taylor au premier ordre d'un champ de vitesse u à divergence nulle autour d'un point x_0 comme

$$u(x) = u(x_0) + \frac{1}{2}\omega(x - x_0)^{\perp} + D(x - x_0) + O(|x - x_0|)$$

où D est une matrice symétrique de trace nulle appelée "matrice de déformation" (pour plus de détails voir [61, Lemme 1.1]).

En prenant le rotationnel de l'Équation (1.1.1), on trouve l'équation d'évolution suivante pour ω :

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ \mathrm{div}(u) = 0 \\ \mathrm{curl}(u) = \omega. \end{cases} \tag{1.1.2}$$

La vorticité est donc transportée par le champ de vitesse : pour u suffisamment régulier (par exemple lipschitzien), ω est constante le long des lignes de champ de u et les normes L^p de ω sont conservées au cours du temps (car u est à divergence nulle).

Précisons également que le champ de vitesse u peut être reconstruit à partir de sa vorticité en résolvant le système elliptique suivant :

$$\begin{cases} \mathrm{div}(u) = 0 \\ \mathrm{curl}(u) = \omega \end{cases}$$

dont la solution est donnée par la loi de Biot-Savart en dimension deux :

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y) \, \mathrm{d}y.$$

Le caractère bien posé des solutions classiques en tout temps des équations d'Euler incompressibles dans le plan a été établi par Wolibner dans [82]. L'existence et l'unicité des solutions faibles a été établi par Yudovich dans [83], qui a démontré le théorème suivant :

Théorème 1.1.1. *Si* $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$*, alors il existe une unique solution faible à l'Équation* (1.1.2) *dans* $L^\infty([0,+\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$.

Ce résultat a été depuis généralisé pour démontrer l'unicité avec des méthodes différentes ou des conditions moins restrictives (par exemple des critères de contrôle de la croissance des normes L^p de ω). Citons notamment [19, 23, 60].

Points vortex

On souhaite décrire la dynamique de petits tourbillons dans un fluide plan, c'est à dire de solutions de l'équation (1.1.2) dont la fonction vorticité est concentrée autour d'un certains nombres de points $(z_1(t), ..., z_N(t))$ (ce qui correspond à un champ de vitesse tournant rapidement au voisinage de ces points).

On cherche donc à obtenir une équation simplifiée pour décrire l'évolution d'une solution de (1.1.2) de la forme :

$$\omega(t) = \sum_{i=1}^{N} a_i \delta_{z_i(t)}$$

où a_i représente l'intensité de masse du tourbillon centré en $z_i(t)$.

La loi de Biot-Savart nous donne le champ de vitesse généré par cette distribution de vorticité :

$$u(t,x) = \sum_{i=1}^{N} \frac{a_i}{2\pi} \frac{(x - z_i(t))^\perp}{|x - z_i(t)|^2}.$$

Comme la vorticité est transportée par u qui est à divergence nulle, on souhaiterait poser $\dot{z}_i(t) = u(t, z_i(t))$. Le champ de vitesse u ci-dessus est cependant singulier au point z_i et la quantité $u(t, z_i(t))$ n'est donc pas bien définie. Si l'on retire ce terme singulier, on obtient le système d'équations suivant

$$\dot{z}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{a_j}{2\pi} \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^2} \tag{1.1.3}$$

qui est appelé "Système des points vortex" ou "loi de Kirchhoff". À cause de la singularité du noyau d'interactions en zéro, ce système est bien posé tant que la distance entre les points reste strictement positive.

Ce calcul formel est justifié rigoureusement par le théorème suivant, prouvé par Marchioro et Pulvirenti dans [63, Chapitre 4, Théorème 4.2] :

Théorème 1.1.2. *On définit*

$$\omega_\varepsilon^0(x) = \varepsilon^{-2} \sum_{i=1}^N a_i \mathbf{1}_{\Lambda_\varepsilon^i}(x)$$

où $a_1, ..., a_N \in \mathbb{R}$ *et* $(\Lambda_\varepsilon^i)_{1 \leqslant i \leqslant N}$ *est une famille d'ouverts disjoints de mesure* ε^2 *et tels que* $\Lambda_\varepsilon^i \subset B(z_i^0, \alpha\varepsilon)$ *pour un certain* $\alpha > 0$. *Soit* $T > 0$ *tel que le système des points vortex* (1.1.3) *soit bien posé sur* $[0, T]$ *(c'est à dire pas de collisions) et* ω_ε *la solution des équations d'Euler issue de* ω_ε^0. *Alors pour tout temps* $t \in [0, T]$ *et toute fonction* f *lisse,*

$$\int_{\mathbb{R}^2} \omega_\varepsilon(t, x) f(x) \, \mathrm{d}x \underset{\varepsilon \to 0}{\longrightarrow} \sum_{i=1}^N a_i f(z_i(t))$$

où $(z_1(t), ..., z_N(t))$ *est la solution de* (1.1.3) *issue de* $(z_1^0, ..., z_N^0)$.

Notons pour finir que les trois quantités suivantes sont des constantes du mouvement : le Hamiltonien

$$H = \sum_{1 \leqslant i \neq j \leqslant N} a_i a_j \ln |x_i - x_j|,$$

le centre de vorticité (pondéré par l'intensité des vortex)

$$M = \sum_{i=1}^N a_i x_i,$$

et le moment d'inertie

$$I = \sum_{i=1}^N a_i |x_i|^2.$$

Une conséquence de la conservation du Hamiltonien et du moment d'inertie est que, pour des intensités toutes du même signe, il n'y a pas de collisions en temps fini. On précise également que pour un bon choix de variables, le Système (1.1.3) peut s'écrire comme le système Hamiltonien associé à H (pour davantage de détails sur ces deux aspects voir [63, Section 4.2]).

1.1.2 Points vortex massiques et système Euler-points vortex massiques

Dans le Chapitre 2 on s'intéressera à la limite de champ moyen d'un modèle décrivant des solides de tailles négligeables plongés dans un fluide plan incompressible non visqueux avec les interactions suivantes : les particules de fluide sont transportées par le champ de vitesse V généré par le fluide et par les particules solides, tandis que les particules solides sont soumises à une force gyroscopique liée à leurs vitesses et à V.

Ce modèle est décrit par les équations suivantes

$$
\begin{cases}
\partial_t \omega + \operatorname{div}(\omega V) = 0 \\
\dot{q}_i = p_i \\
m_i \dot{p}_i = \gamma_i \left(p_i + \nabla^\perp g * \omega(q_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \gamma_j \nabla^\perp g(q_i(t) - q_j(t)) \right)^\perp \\
V = -\nabla^\perp g * (\omega + \rho_N) \\
\rho_N = \sum_{i=1}^{N} \gamma_i \delta_{q_i}
\end{cases} \tag{1.1.4}
$$

où m_i représente la masse du solide i de position q_i, γ_i représente la circulation de fluide autour de ce solide et $g(x) = -\frac{1}{2\pi} \ln |x|$ est l'opposé du noyau du laplacien sur le plan. Remarquons que la force gyroscopique qui s'applique sur les particules est perpendiculaire à la différence de vitesses entre la particule et le fluide environnant. Remarquons également que si l'on fixe $\omega = 0$, alors on retrouve l'équation suivie par des particules chargées interagissant via un potentiel Coulombien répulsif et soumises à un champ magnétique constant :

$$
m_i \dot{p}_i = \gamma_i p_i^\perp - \sum_{\substack{j=1 \\ j \neq i}}^{N} \gamma_i \gamma_j \nabla g(q_i(t) - q_j(t))
$$

où γ_i représente cette fois-ci la charge de la particule q_i.

Ce modèle a été établi par Glass, Lacave, Munnier et Sueur dans [33, 34, 35, 36] en étudiant un ou plusieurs solides plongés dans un fluide et en faisant tendre leurs tailles vers zéro. Le caractère bien posé du système ci-dessus a ensuite été étudié par Lacave et Miot dans [52], qui ont prouvé le théorème suivant (voir [52, Théorème 1.2]) qui établit l'existence de solutions faibles au Système (1.1.4) :

Théorème 1.1.3. *Soient $(q_1^0, ..., q_N^0, p_1^0, ..., p_N^0) \in (\mathbb{R}^2)^{2N}$ tels que $q_i^0 \neq q_j^0$ si $i \neq j$ et $\omega_0 \in L^\infty(\mathbb{R}^2)$ à support compact. Il existe $T^* > 0$ tel que pour tout $0 < T < T^*$ il existe une unique solution faible au système (1.1.4). De plus, si les γ_i ont tous le même signe, alors on peut choisir $T^* = +\infty$.*

On a également un résultat d'unicité, donné par le théorème suivant (voir [52, Théorème 1.5]) :

Théorème 1.1.4. *Si ω_0 est constante au voisinage des points $q_1^0, ..., q_N^0$ alors la solution donnée par le théorème précédent est unique.*

Ces propriétés sont similaires à celles qui avaient déjà été obtenues pour le système mixte Euler-points vortex dans [51, 63, 64].

1.1.3 Modèle de spray gyrocinétiques

On souhaite établir l'équivalent du Système (1.1.4) pour une densité continue de particules en interaction avec un fluide. On considère pour cela le cas de N particules indifférenciées de masses $m_i = N^{-1}$ et de circulations $\gamma_i = N^{-1}$, ce qui donne le système suivant :

$$\begin{cases} \partial_t \omega_N + \operatorname{div}(\omega_N V_N) = 0 \\ \dot{q}_i = p_i \\ \dot{p}_i = \left(p_i^\perp - \nabla g * \omega_N(q_i) - \dfrac{1}{N} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla^\perp g(q_i(t) - q_j(t)) \right) \\ V_N = -\nabla^\perp g * (\omega_N + \rho_N) \\ \rho_N = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \delta_{q_i}. \end{cases} \tag{1.1.5}$$

En prenant (formellement) la limite

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{(q_i(t), p_i(t))} \underset{N \to +\infty}{\longrightarrow} f(t,x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi$$

$$\omega_N \underset{N \to +\infty}{\longrightarrow} \omega$$

on obtient le système suivant

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t f + \xi \cdot \nabla_x f + \operatorname{div}_\xi \Big((\xi - V)^\perp f \Big) = 0 \\ V = -\nabla^\perp g * (\omega + \rho) \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^2} f(t,x,\xi) \, \mathrm{d}\xi. \end{cases} \tag{1.1.6}$$

Ce système a été étudié par Moussa et Sueur dans [69]. Ils ont notamment établi l'existence de solutions faibles et fortes, ainsi que l'unicité dans le cas de densités (ρ, ω) bornées, étendant ainsi les résultats de [83] (pour les équations d'Euler) et de [60] (pour les équations de Vlasov-Poisson). En remplaçant $\nabla^\perp g$ par un noyau d'interactions lipschitzien, ils ont aussi établi la limite de champ moyen du Système (1.1.5) vers le Système (1.1.6) en utilisant des outils de transport optimal à la Dobrushin (voir [25] et la Section 1.2).

Un régime particulier des équations cinétiques (sur lequel se concentrera le Chapitre 2) est le régime hydrodynamique ou monocinétique, c'est à dire le régime où les vitesses des particules solides sont à la même échelle que les vitesses des particules du fluide dans lesquelles elles sont immergées. Plus précisément, on s'intéresse à des distributions de la forme

$$f(t,x,\xi) = \rho(t,x) \otimes \delta_{\xi=v(t,x)}.$$

En insérant cette quantité dans l'Équation (1.1.6), on trouve les équations d'évolutions suivantes pour le système (ω, ρ, v) :

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t v + (v \cdot \nabla) v = (v - V)^\perp \\ V = -\nabla^\perp g * (\omega + \rho). \end{cases} \tag{1.1.7}$$

La stabilité de ce régime a été étudié dans [69] : pour un noyau d'interaction lipschitzien, il est démontré qu'une solution de (1.1.6) proche de $(\omega_0, \rho_0 \otimes \delta_{\xi=v_0(x)})$ reste proche de la solution de (1.1.7) (au sens de la distance de Wasserstein $\mathcal{W}_1$).

Dans le Chapitre 2 (en Section 2.2) on établira le caractère localement bien posé de solutions fortes du Système (1.1.7) (voir la Section 1.3.1 pour plus de détails).

Le résultat principal du Chapitre 2 est le Théorème 2.1.6 qui montre la convergence du système Euler-points vortex massiques (1.1.5) vers le Système (1.1.7). On peut l'énoncer formellement de la façon suivante :

Théorème 1.1.5. *Soit $T > 0$, (ρ, ω, v) une solution faible de (1.1.7) et (ω_N, Q_N, P_N) une solution faible de (1.1.5) (sur l'intervalle $[0, T]$). Sous des hypothèses techniques sur (ω_N, Q_N, P_N) et (ω, ρ, v), et si une certaine fonctionnelle $\mathcal{H}(\omega_0, \rho_0, v_0, \omega_N(0), Q_N(0), P_N(0))$ converge vers zéro, alors pour tout temps $t \in [0, T]$, on a*

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{(q_i(t), p_i(t))} \xrightharpoonup[N \to +\infty]{*} \rho(t) \otimes \delta_{\xi=v(t,x)}$$

pour la convergence faible-$*$ des mesures de probabilités et

$$\|\omega_N(t) - \omega(t)\|_{L^2 \cap \dot{H}^{-1}} \xrightarrow[N \to +\infty]{} 0.$$

On présentera plus en détails ce résultat et la fonctionnelle $\mathcal{H}$ en Section 1.3.1.

1.1.4 Équation des lacs et points vortex pour les lacs

Équation des lacs

L'équation des lacs est un modèle qui décrit l'évolution du champ de vitesse d'un fluide incompressible dans un lac, sous les hypothèses suivantes :

— La profondeur du lac est faible par rapport à l'échelle de variations du champ de vitesse.
— La surface du fluide est presque plate ("petit nombre de Froude").
— La composante verticale du champ de vitesse est faible par rapport à sa composante horizontale.

Ces équations ont notamment été dérivées à partir du modèle de Saint-Venant en rotation rapide par Bresch, Gisclon et Lin dans [8]. Pour une introduction plus générale aux modèles bidimensionnels obtenus à partir de modèles de fluides moyennés dans la direction verticale on renvoie à [38, Chapitre 5].

Les équations des lacs présentent de nombreuses similarités avec les équations d'Euler incompressibles dans le plan, mais prennent en compte les variations de la topographie décrites par une fonction de profondeur b. Lorsque b est constant, on retrouve les équations d'Euler incompressibles dans le plan.

Le caractère bien posé de l'équation des lacs dans des domaines bornés a été étudié par Levermore, Oliver et Titi dans [57]. Dans ce travail les auteurs étudient un analogue du théorème de Yudovich (voir [83]) pour les équations d'Euler incompressibles dans le plan.

Ce résultat a été généralisé plus tard par Bresch et Métivier dans [13] pour prendre en compte le cas où la fonction de profondeur b s'annule au bord, puis par Lacave, Nguyen et Pausader dans [53] pour traiter le cas de fonds irréguliers. L'existence et l'unicité de solutions classiques globales en temps a ensuite été établie par Al Taki et Lacave dans [1].

Dans le Chapitre 3 on étudie un lac infini modélisé par une fonction de profondeur $b : \mathbb{R}^2 \longrightarrow (0, +\infty)$. On s'intéresse à la formulation vorticité de l'équation des lacs :

$$\begin{cases} \partial_t \omega + \operatorname{div}\left(\left(u - \alpha \dfrac{\nabla^\perp b}{b}\right)\omega\right) = 0 \\ \operatorname{div}(bu) = 0 \\ \operatorname{curl}(u) = \omega \end{cases} \tag{1.1.8}$$

où

— $\alpha \in [0, +\infty)$ est un paramètre de forcage,
— $u : [0, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ est le champ de vitesse du fluide.
— $\omega : [0, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ est le champ de vorticité du fluide, défini par

$$\omega = \operatorname{curl}(u) := \partial_1 u_2 - \partial_2 u_1.$$

L'équation des lacs classique ne comporte pas de terme de forcage ($\alpha = 0$). On étudiera cependant ce modèle plus général puisqu'il peut se dériver comme limite de champ moyen du système des points vortex (1.1.10) (dans le régime où les auto-interactions des points ne sont pas négligeables). Ces équations sont un cas particulier d'un modèle étudié par Duerinckx et Fischer dans [27]. Dans cet article les auteurs prouvent le caractère globalement bien posé des solutions faibles et le caractère localement bien posé des solutions fortes de ce modèle.

Points vortex pour les lacs

L'équation des lacs avec forcage (1.1.8) apparaît également comme limite de champ moyen de vortex de type Ginzburg-Landau complexes soumis à des effets d'ancrage et de forcage dans un théorème montré par Duerinckx et Serfaty dans [28]. La dynamique de ces vortex est cependant très différente de la dynamique des tourbillons que l'on peut observer en eaux peu profondes.

Dans le Chapitre 3 on dérivera les Équations (1.1.8) comme limites de champ moyen d'un système d'équations issu d'un modèle introduit par Richardson dans [73]. Dans cet article, en considérant l'équation des lacs ((1.1.8) avec $\alpha = 0$), l'auteur établit par un calcul formel l'équation suivie par le centre de vorticité $q(t)$ d'un vortex dans un lac de profondeur b. À l'ordre principal, cette équation donne

$$\dot{q}(t) \approx -\frac{\Gamma|\ln(\varepsilon)|}{4\pi}\frac{\nabla^{\perp}b(q(t))}{b(q(t))} \tag{1.1.9}$$

où Γ est l'intensité de la vorticité (c'est à dire $\Gamma = \int_{B(q(0),\varepsilon)} \omega$).

Cela signifie qu'à l'ordre principal en ε, un vortex suivra les lignes de topographie constante sans interagir avec les autres vortex restant à distance bornée. L'équation ci-dessus a été rigoureusement justifiée par Dekeyser et Van Schaftingen dans [22] pour le cas d'un seul point vortex. Ce résultat a ensuite été généralisé au cas d'un nombre fini de points vortex par Hientzsch, Lacave et Miot dans [44].

On s'intéresse dans le Chapitre 3 au comportement de N vortex d'intensité N^{-1} lorsque N tends vers $+\infty$. On verra en Section 3.2 que le problème elliptique

$$\begin{cases} \operatorname{div}(bu) = 0 \\ \operatorname{curl}(u) = \omega \end{cases}$$

possède une unique solution donnée par le noyau

$$g_b(x,y) := \sqrt{b(x)b(y)}\,g(x-y) + S_b(x,y)$$

où S_b est la solution d'une équation elliptique (voir l'Équation (3.2.9)) et $g(x) := -\frac{1}{2\pi}\ln|x|$ est l'opposé du noyau du laplacien du plan $\mathbb{R}^2$. Plus précisément, on a

$$u(x) = -\frac{1}{b(x)}\int_{\mathbb{R}^2} \nabla_x^{\perp} g_b(x,y)\omega(y)\,\mathrm{d}y.$$

En utilisant le noyau $\nabla_x^\perp g_b$ on peut donc calculer le champ de vitesse généré par $N-1$ points vortex δ_{q_j} d'intensité $\dfrac{1}{N}$ sur un point vortex δ_{q_i} :

$$-\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla^\perp g_b(q_i,q_j).$$

Ce terme correspond au terme u_{reg} donné par Richardson dans [73, Equation (2.90)]. En combinant cette équation avec le terme d'auto-interaction donné par (1.1.9), on obtient le modèle de points vortex que l'on étudiera dans le Chapitre 3 :

$$\dot{q}_i = -\alpha_N \frac{\nabla^\perp b(q_i)}{b(q_i)} - \frac{1}{Nb(q_i)}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla^\perp g_b(q_i,q_j) \tag{1.1.10}$$

où l'on a noté

$$\alpha_N := \frac{|\ln(\varepsilon_N)|}{4\pi N}$$

avec ε_N la taille des vortex. Dans le cas où $\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$ on étudiera également le régime en temps rapide

$$\overline{q_i}(t) = q_i(\alpha_N^{-1}t). \tag{1.1.11}$$

Remarque 1.1.1. Il n'existe pour l'instant pas de justification rigoureuse des Équations (1.1.10). On ne s'attend pas nécessairement à ce que ces équations décrivent précisément la dynamique d'un nombre fixé de vortex puisque l'on a négligé tous les termes d'auto-interaction d'ordre inférieur à $|\ln(\varepsilon)|$. Cependant le résultat de limite de champ moyen démontré dans le Chapitre 3 (voir Théorème 1.3.3) justifiera rigoureusement que ce modèle simplifié est pertinent dans la limite $N \longrightarrow +\infty$.

Remarque 1.1.2. Il existe plusieurs travaux calculant des trajectoires approchées de vortex pour des profils b spécifiques ainsi que des résultats expérimentaux et numériques liés à la dynamiques des tourbillons dans un lac. Pour plus de détails on renvoie aux résultats de [73] et à la bibliographie associée.

Le résultat principal du Chapitre 3 est le Théorème 3.1.6 qui établit la convergence du système des points vortex (1.1.10) vers l'équation des lacs forcée (1.1.8). On peut l'énoncer formellement de la façon suivante :

Théorème 1.1.6. *Soient ω une solution de (1.1.8) sur $[0,T]$ avec donnée initiale ω_0 et $(q_1,...,q_N)$ une solution de (1.1.10) avec des données initiales bien préparées. Alors si*

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i^0} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \omega_0$$

pour la topologie faible-$*$ des mesures de probabilité et

$$\alpha_N \underset{N\to+\infty}{\longrightarrow} \alpha$$

alors pour tout temps $t \in [0,T]$,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{q_i(t)} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \omega(t)$$

pour la topologie faible-$*$ des mesures de probabilité.

On présentera plus en détails ce résultat dans la Section 1.3.2. Remarquons que dans le cas où $\alpha = 0$, on retrouve l'équation des lacs classique (sans forçage).

Transport selon les lignes de niveau de la topographie

Dans le régime où les auto-interactions sont dominantes (c'est à dire $\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$) le système des points vortex (1.1.10) convergera en temps rapide et pour un grand nombre de particules vers une équation de transport dont les trajectoires sont données par les lignes de niveau de la fonction de profondeur b :

$$\partial_t \overline{\omega} - \operatorname{div}\left(\frac{\nabla^\perp b}{b}\overline{\omega}\right) = 0. \tag{1.1.12}$$

Cela sera montré au Chapitre 3 dans le Théorème 3.1.6, que l'on peut énoncer formellement de la façon suivante :

Théorème 1.1.7. *Soient $\overline{\omega}$ une solution de (1.1.12) sur $[0,T]$ avec donnée initiale $\overline{\omega}_0$ et $(\overline{q_1},...,\overline{q_N})$ une solution de (1.1.11) avec des données initiales bien préparées. Alors si*

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\overline{q_i}^0} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \overline{\omega}_0$$

pour la topologie faible-$*$ des mesures de probabilité et

$$\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$$

alors pour tout temps $t \in [0,T]$,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\overline{q_i}(t)} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \overline{\omega}(t)$$

pour la topologie faible-$*$ des mesures de probabilité.

On présentera plus en détails ce résultat dans la Section 1.3.2.

1.2 Limites de champ moyen

Les limites de champ moyen s'inscrivent dans la famille des problèmes de limites d'échelle en théorie cinétique : on s'intéresse au lien entre des systèmes représentant l'évolution d'un grand nombre de particules (point de vue "microscopique" ou "particulaire" : loi de Newton, loi de Coulomb...), et des systèmes représentant l'évolution d'une densité continue de particules (point de vue "mésoscopique" ou "macroscopique"). Selon la nature des interactions entre les particules, ce type de problème peut concerner des systèmes très différents et donc donner lieu à à des méthodes d'études très variées. Le terme "champ moyen" désigne le cas où les particules sont soumises à un champ de force créé par une moyenne pondérée de forces à longue portée exercées par chacune des autres particules.

1.2.1 Bibliographie

Comme expliqué dans le paragraphe précédent, on cherche à étudier la convergence d'un système d'équations différentielles ordinaires modélisant l'évolution d'un nombre fini de particules $x_1, ..., x_N$ vers la solution d'une équation aux dérivées partielles modélisant l'évolution d'une densité continue de particules, lorsque le nombre de particules tends vers $+\infty$.

Pour des systèmes d'ordre un, on cherche par exemple à étudier la convergence du système

$$\dot{x}_i = \frac{1}{N}\sum_{i=1}^{N} K(x_i - x_j) \tag{1.2.1}$$

vers l'équation de conservation suivante :

$$\partial_t \mu + \mathrm{div}((K * \mu)\mu) = 0 \tag{1.2.2}$$

où $K : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ désigne un certain noyau d'interaction.

Pour des systèmes d'ordre deux on s'intéresse à la convergence d'un système de particules suivant la seconde loi de Newton

$$\ddot{x}_i = \frac{1}{N}\sum_{i=1}^{N} K(x_i - x_j) \tag{1.2.3}$$

vers une équation de type Vlasov modélisant l'évolution d'une densité continue $f(t, x, v)$:

$$\begin{cases} \partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v((K * \mu)f) = 0 \\ \mu(t, x) = \displaystyle\int_{\mathbb{R}^d} f(t, x, v)\,\mathrm{d}v. \end{cases} \tag{1.2.4}$$

Un résultat de limite de champ moyen consiste à prouver que si en temps zéro, la distribution empirique des particules

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)} \qquad \left(\text{respectivement} \quad \frac{1}{N}\sum_{i=1}^{N}\delta_{(x_i(t),\dot{x}_i(t))}\right)$$

converge (en un sens à préciser) vers $\mu(t,x)$ solution de (1.2.2) (respectivement vers $f(t,x,v)$ solution de (1.2.4)), alors la convergence a aussi lieu pour tout temps strictement positif.

Lorsque le noyau d'interactions entre les particules K est lipschitzien, la limite de champ moyen de (1.2.3) vers (1.2.4) a été établie par des arguments de compacité dans [7, 70] ou par des méthodes de transport optimal par Dobrushin dans [25]. Si le noyau K est singulier, d'autres travaux ont étudié des résultats de limites de champ moyen pour des systèmes d'ordre un :

Schochet a montré dans [79] la convergence en champ moyen du système des points vortex (c'est à dire (1.2.1) avec $K(x) = \dfrac{1}{2\pi}\dfrac{x^{\perp}}{|x|^2}$ en dimension 2) vers une solution des équations d'Euler à valeurs dans un espace de mesures, à extraction d'une sous-suite près. Pour cela il a utilisé des arguments développés dans [23] et [78].

Pour des interactions sous-coulombiennes (plus précisément dans le cas où $|K(x)|, |x||\nabla K(x)| \leqslant C|x|^{-\alpha}$ avec $0 < \alpha < d-1$), la limite de champ moyen du Système (1.2.1) a été démontrée par Hauray dans [40] (dans le cas où $\mathrm{div}(K) = 0$) en utilisant des méthodes de transport optimal à la Dobruschin (en s'inspirant des méthodes développées dans [41, 42]). Cette méthode a été réutilisée par Carillo, Choi et Hauray pour étudier la limite de champ moyen de modèles d'aggrégation dans [16].

Dans [26], Duerinckx a montré la limite de champ moyen de systèmes d'ordre un pour des noyaux d'interactions de type Riesz en utilisant une "énergie modulée" introduite par Serfaty dans [80] (voir la quantité définie par (1.2.8)).

Dans [81], Serfaty a utilisé cette méthode d'énergie modulée pour montrer la convergence en champ moyen de systèmes d'ordre un dans le cas où K est donné par un potentiel coulombien, logarithmique ou de type Riesz, ou plus précisément $K = \nabla g$ avec $g(x) = |x|^{-s}$ lorsque $\max(d-2,0) \leqslant s < d$ et $d \geqslant 1$ ou $g(x) = -\ln|x|$ pour $d = 1$ ou 2. Dans cet article $K * \mu$ est supposé lipschitzien.

Rosenzweig a montré dans [75] la limite de champ moyen du système de points vortex sans supposer de régularité lipschitzienne pour le champ de vitesse limite, en utilisant la même énergie modulée que dans [81] avec des estimées améliorées. Remarquons que cela prouve que le système des points vortex converge vers une solution de Yudovich des équations d'Euler (voir [83]). Ce résultat a été étendu au cas de systèmes de dimension supérieure à deux par le même auteur dans [74].

Nguyen, Rosenzweig et Serfaty ont généralisé dans [71] la méthode d'énergie modulée à une classe de potentiels plus généraux, en utilisant des estimées de commutateurs pour des opérateurs de type Calderón-Zygmund.

En s'inspirant de cette méthode d'énergie modulée, Bresch, Jabin et Wang ont défini une "entropie modulée" qui leur a permis d'étudier la limite de champ moyen de particules avec des interactions de type Riesz et du bruit stochastique dans [10, 11, 12]. Cette méthode a été ensuite utilisée pour obtenir de la convergence en champ moyen uniforme en temps par Rosenzweig et Serfaty dans [77] et par Rosenzweig, Serfaty et Chodron de Courcel dans [20].

Le cas des systèmes d'ordre deux est généralement plus complexe, la dérivation comme limite de champ moyen de l'équation de Vlasov-Poisson ((1.2.4) avec $K(x) = \pm\frac{x}{|x|^3}$ en dimension trois) est notamment encore ouverte. Plusieurs travaux ont cependant étudié d'autre cas de limites de champ moyen de systèmes d'ordre deux :

Hauray et Jabin ont traité plusieurs cas d'interactions sous-coulombiennes dans [41, 42] en utilisant des outils de transport optimal à la Dobrushin.

Dans [48, 49], Jabin et Wang ont étudié le cas de gradients bornés et $W^{-1,\infty}$.

Plusieurs travaux ont étudié des limites de champ moyen en régularisant les interactions puis en faisant tendre le paramètre de régularisation vers zéro lorsque N tends vers $+\infty$. Citons par exemple [5, 45, 55, 56].

Dans l'annexe de [81], Duerinckx et Serfaty ont étudié le cas de particules en interaction Coulomb ou Riesz convergeant vers l'équation de Vlasov-Poisson dans le régime monocinétique (c'est à dire l'équation d'Euler-Poisson sans pression) en utilisant l'énergie modulée introduite dans [26, 81].

Dans [39], Han-Kwan et Iacobelli ont utilisé cette même méthode pour étudier la limite de champ moyen de particules en interactions coulombiennes dans un régime quasineutre ou gyrocinétique, vers les équations d'Euler incompressibles. Ce résultat a été généralisé plus tard par Rosenzweig dans [76] pour permettre un choix plus large d'échelles entre le nombre de particules et la constante de couplage.

Bresch, Jabin et Soler ont prouvé dans [9] la dérivation comme limite de champ moyen de l'équation de Vlasov-Fokker-Planck en utilisant la hiérarchie BBGKY et la diffusion en variable de vitesse pour obtenir des estimées sur les marginales.

De nombreux autres travaux ont étudié le cas de particules en interactions avec du bruit, par exemple [4, 6, 17, 30, 48, 49, 54, 58, 71, 72]. Pour une bibliographie plus complète sur les limites de champ moyen dans le cas stochastique, on renvoie à la bibliographie de [20].

Pour une introduction plus générale aux limites de champ moyen on renvoie aux articles de synthèse [37, 47].

1.2.2 Méthodes d'énergie modulée

On présente ici l'idée générale et les grands principes de la méthode d'énergie modulée introduite par Duerinckx dans [26] et Serfaty dans [81] et sur laquelle se basent les travaux de cette thèse. Par simplicité, on se restreint ici à la convergence du système des points vortex

$$\dot{x}_i = -\frac{1}{N}\sum_{i=1}^{N} \nabla^\perp g(x_i - x_j)$$

vers les équations d'Euler (1.1.2) (avec $g(x) = -\frac{1}{2\pi}\ln|x|$).

Montrons tout d'abord comment la structure de l'équation peut être utilisée pour obtenir des estimations de stabilité fort-faible. Les arguments qui suivent sont issus de [26, 80, 81]. Soient ω_1 et ω_2 deux solutions lisses à support compact des équations d'Euler telles que

$$\int_{\mathbb{R}^2} \omega_1 = \int_{\mathbb{R}^2} \omega_2.$$

Notons $u_i = \nabla^\perp g * \omega_i$. La condition précédente permet d'assurer que la quantité

$$\|u_1 - u_2\|_{L^2}^2$$

est bien définie (voir par exemple [61, Proposition 3.3]). En intégrant par parties de façon formelle (voir par exemple la preuve de [81, Proposition 3.3] pour les justifications), on trouve

$$\begin{aligned}\|u_1 - u_2\|_{L^2}^2 &= \int_{\mathbb{R}^2} |\nabla g * (\omega_1 - \omega_2)|^2 \\ &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega_1 - \omega_2)(x)(\omega_1-\omega_2)(y)\,\mathrm{d}x\,\mathrm{d}y \\ &=: \mathcal{F}(\omega_1,\omega_2).\end{aligned} \tag{1.2.5}$$

car $-\Delta g = \delta_0$.

On veut utiliser la quantité $\mathcal{F}(\omega_1,\omega_2)$ pour établir une estimation de stabilité. Calculons donc

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\omega_1,\omega_2) &= -2\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\,\mathrm{div}(u_1\omega_1 - u_2\omega_2)(x)(\omega_1-\omega_2)(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y)\cdot(u_1\omega_1 - u_2\omega_2)(x)(\omega_1-\omega_2)(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

en intégrant par parties. En écrivant

$$u_1\omega_1 - u_2\omega_2 = (u_1-u_2)\omega_1 + u_2(\omega_1-\omega_2)$$

et

$$\int_{\mathbb{R}^2} \nabla g(x-y)(\omega_1-\omega_2)(y)\,\mathrm{d}y = \nabla g * (\omega_1-\omega_2) = (u_1-u_2)^\perp$$

on obtient

(1.2.6)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\omega_1,\omega_2) &= 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} u_2(x)\cdot\nabla g(x-y)(\omega_1-\omega_2)(x)(\omega_1-\omega_2)(y)\,\mathrm{d}x\,\mathrm{d}y\\ &= 2\int_{\mathbb{R}^2} u_2\cdot\nabla\psi\Delta\psi\end{aligned}$$

où l'on a noté $\psi = g*(\omega_1-\omega_2)$. Pour borner l'intégrale obtenue ci-dessus, on va utiliser un outil issu du calcul des variations appelé "tenseur énergie-contraintes", défini par

$$[h,h]_{i,j} := 2\partial_i h\partial_j h - |\nabla h|^2\delta_{i,j}.$$

Pour une introduction plus précise à ce tenseur on réfère à [43, Section 1.3.2]. Si h est suffisamment régulier, on peut montrer par un calcul direct que

$$\mathrm{div}[h,h] = 2\Delta h\nabla h$$

ce qui nous permet d'écrire

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\omega_1,\omega_2) &= \int_{\mathbb{R}^2} u_2\cdot\mathrm{div}([\psi,\psi])\\ &= -\int_{\mathbb{R}^2}\nabla u_2 : [\psi,\psi]\end{aligned} \tag{1.2.7}$$

en intégrant par parties. Comme $[\psi,\psi]$ est une expression quadratique des dérivées de ψ, on obtient

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\omega_1,\omega_2)\right| \leqslant C\,\|\nabla u_2\|_{L^\infty}\,\|\nabla\psi\|_{L^2}^2$$

et

$$\|\nabla\psi\|_{L^2}^2 = \|\nabla g*(\omega_1-\omega_2)\|_{L^2}^2 = \mathcal{F}(\omega_1,\omega_2).$$

Par lemme de Grönwall on obtient donc, pour $t\leqslant T$,

$$\mathcal{F}(\omega_1(t),\omega_2(t)) \leqslant e^{C\|\nabla u_2\|_{L^1([0,T],L^\infty)}}\mathcal{F}(\omega_1(0),\omega_2(0)).$$

Remarquons que seule la norme de u_2 intervient dans l'inégalité précédente : on peut donc généraliser cette inégalité à des fonctions ω_1 moins régulières, et ainsi obtenir un principe de stabilité fort-faible $\dot{H}^{-1}$ pour les équations d'Euler.

On souhaiterait utiliser le même principe pour étudier la convergence du système des points vortex. Cependant, la quantité

$$\omega_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$$

n'est pas dans H^{-1} et la quantité $\mathcal{F}(\omega_N, \omega)$ n'est donc pas définie. On peut cependant observer que le seul problème de définition de cette quantité vient de la singularité de g sur la diagonale $\Delta = \{(x,x) \mid x \in \mathbb{R}^2\}$. On peut donc la supprimer de l'intégrale pour définir
(1.2.8)

$$\mathcal{F}(X_N, \mu) := \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\,\mathrm{d}\left(\mu - \sum_{i=1}^N \delta_{x_i}\right)(x)\,\mathrm{d}\left(\mu - \sum_{i=1}^N \delta_{x_i}\right)(y).$$

Cette quantité est l'énergie modulée introduite par Duerinckx dans [26] et Serfaty dans [81] pour démontrer la limite de champ moyen de plusieurs systèmes à interactions singulières. Une partie importante des résultats de ces travaux consiste à démontrer que cette quantité est adaptée pour mesurer la distance entre une mesure empirique de points et une densité continue. On a notamment le résultat de coercivité suivant (voir par exemple [75, Proposition 3.10] ou [81, Proposition 3.6]) :

Proposition 1.2.1. *Soient $s < -1$ et $x_1, ..., x_N$ tels que $x_i \neq x_j$ si $i \neq j$ et $\mu \in L^\infty$ une densité de probabilité à support compact, alors*

$$\left\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\right\|_{H^s} \leqslant C_s(|\mathcal{F}(X_N,\mu)|^{\frac{1}{2}} + N^{-\frac{1}{2}}|\ln(N)|^{\frac{1}{2}} + (1+\|\mu\|_{L^\infty})N^{-\frac{1}{2}}))$$

Expliquons brièvement le principe de la preuve de cette proposition. Soit $\eta > 0$ un petit paramètre et $\delta_{x_i}^{(\eta)}$ une régularisation de la masse de dirac δ_{x_i}. Alors on peut montrer que

$$\begin{aligned}\mathcal{F}(X_N,\mu) &= \int_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\,\mathrm{d}\left(\mu - \sum_{i=1}^N \delta_{x_i}^{(\eta)}\right)(x)\,\mathrm{d}\left(\mu - \sum_{i=1}^N \delta_{x_i}^{(\eta)}\right)(y) + o(\eta)\\ &= \int_{\mathbb{R}^2}\left|\nabla g * \left(\mu - \sum_{i=1}^N \delta_{x_i}^{(\eta)}\right)\right|^2 + o(\eta)\end{aligned}$$

en intégrant par parties comme on l'a fait pour (1.2.5). Notons maintenant

$$H_{N,\eta}^{\mu} = g * \left(\mu - \sum_{i=1}^N \delta_{x_i}^{(\eta)}\right)$$

le potentiel associé à la distribution régularisée. Pour ξ lisse à support compact on a alors

$$\begin{aligned}\int_{\mathbb{R}^2} \xi \left(\mu - \sum_{i=1}^N \delta_{x_i}\right) &= \int_{\mathbb{R}^2} \xi \left(\mu - \sum_{i=1}^N \delta_{x_i}^{(\eta)}\right) + o(\eta) \\ &= -\int \xi \Delta H_{N,\eta}^{\mu} + o(\eta) \\ &\leqslant \|\nabla \xi\|_{L^2} \left\|\nabla H_{N,\eta}^{\mu}\right\|_{L^2} + o(\eta) \\ &\leqslant \|\nabla \xi\|_{L^2} \mathcal{F}(X_N, \mu) + o(\eta).\end{aligned}$$

En choisissant une bonne suite $\eta_N \underset{N\to+\infty}{\longrightarrow} 0$ et en étant plus précis et explicite sur les restes que l'on a noté $o(\eta)$ (qui vont faire apparaitre des termes négligeables dépendant de N et d'autres normes de ξ), on obtient la preuve de la Proposition 1.2.1 par dualité.

Il a été également montré par Duerinckx dans [26, Lemme 2.6] que la convergence en énergie modulée correspond à demander la convergence faible-$*$ de la mesure empirique de points vortex ainsi que la convergence de l'énergie associée au système :

Proposition 1.2.2. *Soient $x_1, ..., x_N$ tels que $x_i \neq x_j$ si $i \neq j$ et $\mu \in L^\infty$ une densité de probabilité à support compact, alors les deux assertions suivantes sont équivalentes*

1. $\mathcal{F}(X_N, \mu) \underset{N\to+\infty}{\longrightarrow} 0.$
2. $\displaystyle\frac{1}{N}\sum_{i=1}^N \delta_{x_i} \underset{N\to+\infty}{\longrightarrow} \mu$ *pour la topologie faible-$*$ de l'espace des mesures de probabilités et*

$$\frac{1}{N^2} \sum_{1\leqslant i\neq j\leqslant N} g(x_i - x_j) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\mu(x)\mu(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Pour plus de détails sur ces hypothèses on renvoie à l'introduction de [26].

Remarquons enfin que la quantité $\mathcal{F}$ n'est plus une distance à proprement parler puisqu'elle n'est plus nécessairement positive. On a cependant le résultat de positivité asymptotique suivant (voir [81, Corollaire 3.5]) :

Proposition 1.2.3. *Soient $x_1, ..., x_N$ tels que $x_i \neq x_j$ si $i \neq j$ et $\mu \in L^\infty$ une densité de probabilité à support compact, alors*

$$\mathcal{F}(X_N, \mu) \geqslant -\frac{1}{2}\ln(N)N^{-1} - CN^{-1}.$$

Les trois propositions ci-dessus nous permettent d'assurer que la quantité $\mathcal{F}$ est une bonne mesure de la distance entre une distribution empirique de points vortex $(x_1(t), ..., x_N(t))$ et une densité de vorticité régulière $\omega(t, x)$. On veut donc se servir de cette quantité pour établir une estimation de type Grönwall comme on l'a fait plus haut pour deux solutions régulières des équations d'Euler. Le calcul de la dérivée est similaire à celui qui a été effectué en (1.2.6) et on peut démontrer que

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(X_N(t), \omega(t)) =& 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} u(t,x)\cdot\nabla g(x-y) \\ &\times \mathrm{d}\left(\omega(t) - \sum_{i=1}^N \delta_{x_i(t)}\right)(x)\,\mathrm{d}\left(\omega(t) - \sum_{i=1}^N \delta_{x_i(t)}\right)(y).\end{aligned}$$

Pour pouvoir borner le terme ci-dessus, on va utiliser de nouveau des régularisations des masses de dirac $\delta_{x_i}^{(\eta)}$.

$$\begin{aligned}&\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} u(x)\cdot\nabla g(x-y)\,\mathrm{d}\left(\sum_{i=1}^N \delta_{x_i} - \omega\right)^{\otimes 2}(x,y) \\ &\quad = \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} u(x)\cdot\nabla g(x-y)\,\mathrm{d}\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - \omega\right)^{\otimes 2}(x,y) + o(\eta) \\ &\quad \leqslant C\int_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\,\mathrm{d}\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - \omega\right)(x)\,\mathrm{d}\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - \omega\right)(y) \\ &\quad + o(\eta) \\ &\quad \leqslant C\mathcal{F}(X_N(t), \omega(t)) + o(\eta)\end{aligned}$$

où on a utilisé le tenseur $[\cdot,\cdot]$ comme en (1.2.7). En étant suffisamment précis sur le contrôle des restes et en choisissant $\eta = N^{-\alpha}$ pour un α bien choisi, on obtient une estimation du type

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(X_N(t), \omega(t))\right| \leqslant C(\mathcal{F}(X_N(t), \omega(t)) + N^{-\beta})$$

pour $\beta > 0$ indépendant de u et ω, et C dépendant de certaines normes de ω (pour une preuve et un énoncé plus précis, voir [81, Proposition 1.1]). Par le lemme de Grönwall on obtient donc

$$\mathcal{F}(X_N(t), \omega(t)) \leqslant C(\mathcal{F}(X_N(0), \omega(0)) + N^{-\beta})$$

et il découle directement de la Proposition 1.2.2 le théorème suivant

Théorème 1.2.1. *Soit ω solution d'Euler suffisamment régulière et $X_N \in (\mathbb{R}^2)^N$, alors si*

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i(0)} \xrightharpoonup[N\to+\infty]{*} \omega(0)$$

et

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g(x_i(0)-x_j(0)) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\omega(0,x)\omega(0,y)\,\mathrm{d}x\,\mathrm{d}y$$

alors pour tout $t>0$*,*

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)} \underset{N\to+\infty}{\overset{*}{\longrightarrow}} \omega(t)$$

et

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g(x_i(t)-x_j(t)) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\omega(t,x)\omega(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Précisons que cette méthode s'applique à des cas plus généraux que le système des points vortex : des noyaux d'interactions plus ou moins singuliers que le noyau de Coulomb, des systèmes avec du bruit, etc. (Pour plus de détails voir la Section 1.2.1.)

Concluons cette section en expliquant comment la méthode d'énergie modulée détaillée ci-dessus peut être utilisée pour étudier des systèmes d'ordre deux. Considérons par exemple le système :

$$\ddot{x}_i = -\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(x_i-x_j). \tag{1.2.9}$$

Comme expliqué en Section 1.2.1, la limite de champ moyen du système ci-dessus vers l'équation de Vlasov-Poisson

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f + (\nabla g * \rho)\cdot\nabla_\xi f = 0 \\ \rho(t,x) = \displaystyle\int f(t,x,\xi)\,\mathrm{d}\xi \end{cases}$$

n'est pas encore justifiée rigoureusement. Concentrons nous sur le cas particulier de distributions monocinétiques, c'est à dire de la forme $f(t,x,\xi) = \rho(t,x)\otimes\delta_{\xi=v(t,x)}$. En insérant formellement cette quantité dans l'équation ci-dessus, on trouve un système couplé sur (ρ,v) appelé "Équations d'Euler-Poisson sans pression" :

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho v) = 0 \\ \partial_t v + (v\cdot\nabla)v = -\nabla g * \rho. \end{cases}$$

Dans ce régime, Duerinckx et Serfaty ont démontré dans l'annexe de [81] la limite de champ moyen du Système (1.2.9) vers les équations d'Euler-Poisson sans pression à l'aide de l'énergie modulée suivante :

$$\mathcal{H}^c = \frac{1}{N}\sum_{i=1}^{N}|\dot{x}_i - v(t,x_i)|^2 + \mathcal{F}(X_N,\rho). \tag{1.2.10}$$

C'est cette méthode qui sera adaptée dans le Chapitre 2 pour étudier la limite de champ moyen du Système (1.1.5).

1.3 Plan et arguments de preuves

Dans cette section on présente le plan du manuscrit et les différents arguments de preuves utilisés dans cette thèse.

Dans le Chapitre 2 on étudie le modèle de spray gyrocinétique présenté en Section 1.1.3. On montre tout d'abord le caractère localement bien posé des solutions fortes. On prouve ensuite la limite de champ moyen du système vortex wave gyrocinétique présenté en Section 1.1.2 vers le modèle de spray.

Dans le Chapitre 3 on étudie la limite de champ moyen du modèle de points vortex pour l'équation des lacs (1.1.10).

1.3.1 Étude du modèle de spray gyrocinétique

Caractère localement bien posé en temps du modèle de spray

Le caractère localement bien posé des équations d'Euler Poisson (c'est à dire le Système (1.1.7) avec $\omega = 0$ et un terme de pression dépendant de ρ dans l'équation sur v) a été étudié par Makino dans [62] pour le cas $d = 3$ en utilisant les estimées hyperboliques prouvées dans [50]. Dans la Section 2.2, on étendra ce résultat au cas du Système (1.1.7). Le théorème peut s'énoncer de la façon suivante (pour un énoncé plus précis, voir le Théorème 2.2.1) :

Théorème 1.3.1. *Soit $s \geqslant 3$ un entier, $u_0 \in H^{s+1}(\mathbb{R}^2, \mathbb{R}^2)$ et $\rho_0, \omega_0 \in H^s(\mathbb{R}^2, \mathbb{R})$ à supports compacts, alors si T est suffisamment petit il existe un unique $(u + \overline{V}, \omega, \rho)$ solution de (1.1.7) avec condition initiale $(u_0 + \overline{V}, \omega_0, \rho_0)$ (où $\overline{V}$ est un champ de vecteur lisse indépendant du temps tel que $\int_{\mathbb{R}^2} \operatorname{curl}(\overline{V}) = \int_{\mathbb{R}^2} (\rho_0 + \omega_0)$).*

La preuve de ce théorème repose sur une méthode de point fixe que l'on présente plus en détails dans l'introduction de la Section 2.2.

Ces solutions donneront une classe d'exemples sur lesquelles pourra être appliqué le théorème de limite de champ moyen 2.1.6 que l'on présente dans la prochaine section.

Limite de champ moyen du système Euler-points vortex massiques

Dans la Section 2.3 on montre la convergence du Système (1.1.5) vers (1.1.7) dans le régime monocinétique en adaptant la méthode développée par Duerinckx et Serfaty dans [81] que l'on a présenté en fin de Section 1.2.2.

Pour cela on va introduire une énergie modulée similaire à celle qui a été définie en (1.2.10) mais qui prendra en compte les vorticités ω_N et ω

des fluides dans lesquels sont plongés les solides dans les modèles (1.1.5) et (1.1.7).

Soient $\rho, \omega, \omega_N \in (L^1 \cap L^\infty)(\mathbb{R}^2, \mathbb{R})$ à support compact, $v \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ et $Q_N, P_N \in (\mathbb{R}^2)^N$ tels que $q_i \neq q_j$ si $i \neq j$. Supposons que

$$\int_{\mathbb{R}^2} \omega = \int_{\mathbb{R}^2} \omega_N$$
$$\int_{\mathbb{R}^2} \rho = 1.$$

On définit :
(1.3.1)

$$\begin{aligned}\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N) \\ :=& \frac{1}{N} \sum_{i=1}^{N} |v(q_i) - p_i|^2 \\ &+ \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta} g(x-y)(\rho + \omega - \rho_N - \omega_N)^{\otimes 2}(\mathrm{d}x\, \mathrm{d}y) \\ &+ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x-y)(\omega - \omega_N)(x)(\omega - \omega_N)(y)\, \mathrm{d}x\, \mathrm{d}y \\ &+ \|\omega - \omega_N\|_{L^2}^2 + BN^{-\gamma}\end{aligned}$$

où

$$\rho_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{q_i}$$

et où γ et B sont des constantes strictement positives qui permettent d'assurer que la quantité $\mathcal{H}$ est positive (ce qui sera prouvé dans la Proposition 2.1.5).

Le théorème principal de cette section est le suivant (pour un énoncé précis voir le Théorème 2.1.6) :

Théorème 1.3.2. *Soit $T > 0$, (ρ, ω, v) une solution faible de (1.1.7) et (ω_N, Q_N, P_N) une solution faible de (1.1.5) (sur l'intervalle $[0, T]$). Supposons que*

$$\int_{\mathbb{R}^2} \omega_{N,0} = \int_{\mathbb{R}^2} \omega_0$$
$$\int_{\mathbb{R}^2} \rho_0 = 1.$$

On définit

$$\mathcal{H}_N(t) := \mathcal{H}(\omega(t), \rho(t), v(t), \omega_N(t), Q_N(t), P_N(t)).$$

Si l'on suppose que $(q_1(0), ..., q_N(0))$ sont deux à deux distincts et n'intersectent pas le support de ω_N^0, et sous des hypothèses de régularité sur

(ω, ρ, v) *il existe des constantes* C *and* β *dépendant seulement de* T, ρ, ω *et de* $\sup_N \|\omega_N\|_{L^1 \cap L^\infty}$ *telles que pour tout* $t \in [0, T]$,

$$\mathcal{H}_N(t) \leqslant C(\mathcal{H}_N(0) + N^{-\beta}).$$

La convergence de $\mathcal{H}_N(0)$ vers zéro implique donc la convergence de $\mathcal{H}_N(t)$ vers zéro pour tout temps $t \in [0, T]$.

Pour préciser la notion de convergence induite par la fonctionnelle $\mathcal{H}$ on montrera également un résultat de coercivité (pour un énoncé précis voir la Proposition 2.1.9) :

Proposition 1.3.1. *Soient* $Q_N, P_N \in (\mathbb{R}^2)^N$ *et* $\omega, \omega_N, \rho \in L^1 \cap L^\infty(\mathbb{R}^2, \mathbb{R})$ *à support compacts et* $v \in W^{2,\infty}(\mathbb{R}^2, \mathbb{R}^2)$*. Supposons*

$$\int_{\mathbb{R}^2} \omega = \int_{\mathbb{R}^2} \omega_N$$
$$\int_{\mathbb{R}^2} \rho = 1.$$

et

$$\sup_{N \in \mathbb{N}} \|\omega_N\|_{L^1 \cap L^\infty} < +\infty.$$

Alors il existe des constantes strictement positives C *et* β *telles que*

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \delta_{(q_i, p_i)} - \rho \otimes \delta_{\xi = v(x)} \right\|_{H^{-5}} \leqslant C(1 + \|\nabla v\|^2_{W^{1,\infty}}) \times (\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N)^{\frac{1}{2}} + (1 + \|\rho\|_{L^\infty}) N^{-\beta})^{\frac{1}{2}}.$$

En particulier, si on suppose

$$\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N) \underset{N \to \infty}{\longrightarrow} 0$$

alors pour tout $a < -1$,

$$\rho_N \underset{N \to +\infty}{\longrightarrow} \rho \ \text{ dans } H^a$$
$$\omega_N - \omega \underset{N \to +\infty}{\longrightarrow} 0 \ \text{ dans } L^2 \cap \dot{H}^{-1}$$
$$\frac{1}{N} \sum_{i=1}^{N} \delta_{(q_i, p_i)} \underset{N \to +\infty}{\longrightarrow} \rho \otimes \delta_{\xi = v(x)} \ \text{ dans } H^{-5}.$$

Remarquons pour finir que si l'on suppose la convergence de $\omega_{N,0} - \omega_0$ vers zéro dans $\dot{H}^{-1}$ et la convergence de $\rho_{N,0}$ vers ρ_0 pour la topologie faible-$*$ des mesures de probabilités, alors la convergence de

$$\int_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta} g(x - y)(\rho_0 + \omega_0 - \rho_{N,0} - \omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)$$

vers zéro peut-être vue comme une condition de bonne préparation des données initiales, comme on le montrera dans la proposition suivante (voir la Proposition 2.1.12), dont la preuve sera une conséquence directe de la Proposition 1.2.2 établie par Duerinckx dans [26].

Proposition 1.3.2. *Soient $\omega_{N,0}, \omega_0, \rho_0 \in L^\infty$ à supports compacts telles que $\int \omega_{N,0} = \int \omega_0$, $\int \rho_0 = 1$ et soient $(q_1^0, ..., q_N^0)$ tels que $q_i^0 \neq q_j^0$ si $i \neq j$. Supposons*

$$\int_{\mathbb{R}^2} \omega_0 = \int_{\mathbb{R}^2} \omega_{N,0},$$
$$\int_{\mathbb{R}^2} \rho_0 = 1,$$

$$\sup_{N\in\mathbb{N}} \left\|\omega_N^0\right\|_{L^1\cap L^\infty} < +\infty$$

et

$$\omega_{N,0} - \omega_0 \underset{N\to+\infty}{\longrightarrow} 0 \quad \text{dans } \dot{H}^{-1}$$
$$\rho_{N,0} \underset{N\longrightarrow+\infty}{\overset{*}{\rightharpoonup}} \rho_0 \quad \text{dans } \mathcal{M}(\mathbb{R}^2)$$
$$\frac{1}{N^2} \sum_{1\leqslant i\neq j\leqslant N} g(q_i^0 - q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint g(x-y)\rho_0(x)\rho_0(y)\,\mathrm{d}x\,\mathrm{d}y$$

alors

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho_0 + \omega_0 - \rho_{N,0} - \omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \underset{N\to+\infty}{\longrightarrow} 0.$$

Par des arguments de compacité que l'on détaillera dans l'introduction du Chapitre 2, les propositions ci-dessus permettent d'assurer que sous les hypothèses du Théorème 1.3.2, on a

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{(q_i(t),p_i(t))} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \rho(t)\otimes\delta_{\xi=v(t,x)} \quad \text{dans } \mathcal{M}(\mathbb{R}^2)$$

et

$$\|\omega_N(t) - \omega(t)\|_{L^2\cap\dot{H}^{-1}} \underset{N\to+\infty}{\longrightarrow} 0$$

ce qui donne la convergence en champ moyen du Système (1.1.5) vers le Système (1.1.7).

1.3.2 Limite de champ moyen du système des points vortex pour l'équation des lacs

Le résultat principal du Chapitre 3 est le théorème suivant, qui donne la limite de champ moyen du système des points vortex (1.1.10) et de sa version renormalisée en temps (1.1.11) (pour un énoncé précis, voir le Théorème 3.1.6). Rappelons que g_b (qui sera défini en (3.2.10)) désigne le noyau du problème elliptique $-\operatorname{div}\left(\frac{1}{b}\nabla\psi\right) = \omega$. Plus précisément, on a

$$\psi(x) = \int_{\mathbb{R}^2} g_b(x,y)\omega(y)\,\mathrm{d}y.$$

On rappelle également que l'on a noté $I(Q_N) = \frac{1}{N}\sum_{i=1}^{N}|q_i|^2$.

Théorème 1.3.3. *Soient ω une solution de (1.1.8) issue de ω_0 et $\overline{\omega}$ une solution de (1.1.12) issue de ω_0. Supposons que ω, $\overline{\omega}$ et b sont suffisamment régulières. On a la convergence en champ moyen du système des points vortex dans les deux régimes suivants :*

1. *Supposons que :*
   - $(I(Q_N(0)))_N$ *est bornée.*
   - $\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i^0} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \omega_0$ *pour la topologie faible-$*$ des mesures de probabilité.*
   - $\alpha_N \underset{N\to+\infty}{\longrightarrow} \alpha$.
   - $\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i^0,q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_0(x)\omega_0(y)\,\mathrm{d}x\,\mathrm{d}y.$

   Alors pour tout $t\in[0,T]$, $\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i(t)} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \omega(t)$ pour la topologie faible-$*$ des mesures de probabilité et

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i(t),q_j(t)) \longrightarrow \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(t,x)\omega(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

2. *Supposons que*
   - $(I(Q_N(0)))_N$ *est bornée.*
   - $\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i^0} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \omega_0$ *pour la topologie faible-$*$ des mesures de probabilité.*
   - $\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$.

— $\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i^0,q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_0(x)\omega_0(y)\,\mathrm{d}x\,\mathrm{d}y.$

Alors pour tout $t\in[0,T]$, $\frac{1}{N}\sum_{i=1}^N \delta_{\overline{q_i}(t)} \underset{N\to+\infty}{\overset{*}{\longrightarrow}} \overline{\omega}(t)$ *pour la topologie faible-$*$ des mesures de probabilité et*

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(\overline{q_i}(t),\overline{q_j}(t)) \longrightarrow \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\overline{\omega}(t,x)\overline{\omega}(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Remarquons que dans le cas $\alpha_N \underset{N\to+\infty}{\longrightarrow} 0$ on retrouve l'équation des lacs usuelle ((1.1.8) avec $\alpha=0$).

On demande à $(I(Q_N(0)))_N$ d'être bornée pour garantir qu'il n'y a pas trop de fuite de vorticité à l'infini. Cette hypothèse n'était pas nécessaire dans les travaux de Serfaty et Duerinckx [26, 81] mais nous sera utile pour traiter l'hétérogénéité du noyau g_b.

Comme pour ce qui a été décrit pour la Proposition 1.2.2, la convergence de l'énergie et la convergence faible-$*$ de ω_N vers ω assurent la convergence de ω_N vers ω en un sens plus fort : on prouvera au Corollaire 1.3.4 que sous certaines hypothèses, cela est équivalent à la convergence vers zéro de "l'énergie modulée des lacs", définie par :

$$\text{(1.3.2)}\quad \mathcal{F}_b(Q_N,\omega) := \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i}-\omega\right)(x)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i}-\omega\right)(y)$$

où $(q_1,\dots,q_N)\in(\mathbb{R}^2)^N$, $\omega\in L^\infty$ à support compact, g_b est le noyau défini en (3.2.10) et

$$\Delta := \{(x,x)\ ;\ x\in\mathbb{R}^2\}.$$

On utilisera cette énergie pour mesurer la distance entre les solutions ω et Q_N de (1.1.8) et (1.1.10) ou les solutions $\overline{\omega}$ et $\overline{Q_N}$ de (1.1.12) et (1.1.11).

Le Chapitre 3 est organisé comme suit :

Dans la Section 3.2 on démontre le caractère bien posé du problème elliptique liant un champ de vitesse satisfaisant $\mathrm{div}(bu)=0$ à sa vorticité, on construit le noyau de Green g_b de ce problème elliptique et on établit plusieurs estimations liées à ce noyau. On se base notamment sur des résultats établis par McOwen dans [65, 66] et par Duerinckx dans [27].

Dans la Section 3.3 on montre que le système des points vortex est bien posé et on établit des bornes sur l'énergie et le moment d'inertie du système dont on aura besoin en Section 3.7.

Dans la Section 3.4 on calcule la dérivée en temps des énergies modulées.

Dans la Section 3.5 on établit plusieurs propriétés de la fonctionnelle $\mathcal{F}_b$. On montre notamment le résultat de coercivité suivant (pour un énoncé précis voir le Corollaire 3.5.3) :

Corollaire 1.3.3. *Il existe $\beta > 0$ tel que pour tout $s < -1$, si ω est une densité de probabilité bornée à support compact et Q_N est tel que $q_i \neq q_j$ si $i \neq j$, alors sous des hypothèses de régularité sur b, on a*

$$\left\| \frac{1}{N}\sum_{i=1}^{N}\delta_{q_i} - \omega \right\|_{H^s} \leqslant C_b((1 + I(Q_N) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty})N^{-\beta} + \mathcal{F}_b(\omega, Q_N)).$$

On établit également que la convergence de $\mathcal{F}_b$ vers zéro est équivalente (modulo certaines hypothèses techniques) à avoir convergence faible-$*$ de la distribution empirique du système des points vortex ainsi que la convergence de l'énergie associée (voir le Corollaire 3.5.4 pour un énoncé précis) :

Corollaire 1.3.4. *Soit ω est une densité de probabilité bornée à support compact, Q_N tel que $q_i \neq q_j$ si $i \neq j$. Supposons que $(I(Q_N))_N$ est bornée, alors sous des hypothèses de régularité sur b, les deux assertions suivantes sont équivalentes :*

1. $\mathcal{F}_b(\omega, Q_N) \underset{N\to+\infty}{\longrightarrow} 0.$
2. $\displaystyle\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i} \underset{N\to+\infty}{\longrightarrow} \omega$ *pour la topologie faible-$*$ des mesures de probabilités et*

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i, q_j) \longrightarrow \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Dans la Section 3.6 on borne le terme principal apparaissant dans les dérivées calculées en Section 3.4.

Dans la Section 3.7 on utilise les résultats des autres sections pour démontrer le Théorème 1.3.3.

L'énergie modulée $\mathcal{F}_b$ est similaire à celle définie par (1.2.8) et les preuves des Sections 3.4 à 3.7 suivent le mêmes idées que celles expliquées en Section 1.2.2 : calculer la dérivée de l'énergie modulée, régulariser les masses de dirac et contrôler les termes de restes dans les différents termes qui apparaissent dans ces dérivées puis appliquer un lemme de Grönwall.

Chapitre 2

Mean-field limit derivation of a spray model with gyroscopic effects

The results of this chapter are the content of [67].

2.1 Introduction

The purpose of this chapter is to establish the mean-field limit of the gyrokinetic vortex-wave system introduced in Section 1.1.2. We recall that it models the evolution of N solid particles with positions $q_1, ..., q_N$ and velocities $p_1, ..., p_N$ in a fluid of vorticity ω_N:

$$\left\{\begin{aligned} &\partial_t \omega_N + \operatorname{div}(\omega_N V_N) = 0 \\ &\dot{q}_i = p_i \\ &\dot{p}_i = p_i^\perp - \nabla g * \omega_N(q_i) - \frac{1}{N} \sum_{\substack{j=1 \\ j\neq i}}^N \nabla g(q_i(t) - q_j(t)) \\ &V_N = -\nabla^\perp g * (\omega_N + \rho_N) \\ &\rho_N = \frac{1}{N} \sum_{k=1}^N \delta_{q_k} \end{aligned}\right. \tag{2.1.1}$$

where $\dfrac{1}{N}$ represents both the mass of a solid particle and the circulation of velocity around it and $g(x) := -\dfrac{1}{2\pi} \ln |x|$.

In the monokinetic regime, when the number of particles becomes very large, we want to recover the following partial differential equations (introduced in Section 1.1.3)

$$
\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t v + (v \cdot \nabla) v = (v - V)^{\perp} \\ V = -\nabla^{\perp} g * (\omega + \rho). \end{cases} \tag{2.1.2}
$$

Before establishing the mean-field limit, we will justify the local in time existence and uniqueness of strong solutions of System (2.1.2). The local well-posedness of Euler-Poisson system (that is the system we get if we take $\omega = 0$ and add a pressure term in the equation on v) was studied in [62] in the case $d = 3$ using the usual estimates on hyperbolic systems that were proved in [50]. In Section 2.2 we extend this result to System (2.1.2).

2.1.1 Main results

The main result in this paper is Theorem 2.1.6 which proves the mean-field limit of solutions of System (2.1.1) to solutions of (2.1.2) with some regularity assumptions. We will use the following definition of weak solutions:

Definition 2.1.1. *We say that (ρ, ω, v) is a weak solution of (2.1.2) if*

1. *$\rho, \omega \in \mathcal{C}^0([0,T], L^1 \cap L^\infty(\mathbb{R}^2, \mathbb{R}))$ with compact supports.*
2. *For all $t \in [0,T]$, $\displaystyle\int_{\mathbb{R}^2} \rho(t) = 1$.*
3. *$v \in W^{1,\infty}([0,T] \times \mathbb{R}^2, \mathbb{R}^2)$*
4. *The equation on the velocity is satisfied almost everywhere and the continuity equations are satisfied in the sense of distributions, that is for every $\varphi \in W^{1,\infty}([0,T], \mathcal{C}^1(\mathbb{R}^2))$ with compact support and for every $t \in [0,T]$, we have:*

$$
\begin{aligned} \int_{\mathbb{R}^2} (\rho(t)\varphi(t) - \rho_0 \varphi(0)) &= \int_0^t \int_{\mathbb{R}^2} \rho(s,x)(\partial_s \varphi + \nabla \varphi \cdot v)(s,x) \, \mathrm{d}x \, \mathrm{d}s \\ \int_{\mathbb{R}^2} (\omega(t)\varphi(t) - \omega_0 \varphi(0)) &= \int_0^t \int_{\mathbb{R}^2} \omega(s,x)(\partial_s \varphi + \nabla \varphi \cdot V)(s,x) \, \mathrm{d}x \, \mathrm{d}s \end{aligned} \tag{2.1.3}
$$

Remark that by conservation of mass it is enough to ask

$$
\int_{\mathbb{R}^2} \rho_0 = 1
$$

to get Assumption (2).

In Section 2.2 we will prove existence and uniqueness of solutions of (2.1.2) in a space strictly included in $\mathcal{C}^0([0,T], L^1 \cap L^\infty)^2 \times W^{1,\infty}$ (see Theorem 2.2.1). For the microscopic system (2.1.1), we will use the following definition of weak solutions, introduced in [52]:

Definition 2.1.2. *(ω_N, Q_N, P_N) is a weak solution of (2.1.1) on $[0,T]$ if*

1. *$\omega_N \in L^\infty([0,T], L^1 \cap L^\infty) \cap \mathcal{C}^0([0,T], L^\infty - w^*)$ with compact support.*
2. *$q_1, ..., q_N \in \mathcal{C}^2([0,T], \mathbb{R}^2)$*
3. *The partial differential equation on ω_N is satisfied in the sense of distributions (which means that it also verifies (2.1.3)) and the ordinary differential equations are satisfied in the classical sense.*

Remark 2.1.3. By Theorems 1.4 and 1.5 of [52] we know that for $\omega_N(0) \in L^\infty(\mathbb{R}^2)$ compactly supported and $q_1(0), ..., q_N(0)$ distinct outside of the support of $\omega_N(0)$ there exists a unique weak solution of (2.1.1) on $[0,T]$ for any $T > 0$ and no collision between the solid particles occurs in finite time. It follows by [52, Corollary A.2] that for all $1 \leqslant p \leqslant \infty$, $\|\omega_N(t)\|_{L^p} = \|\omega_N(0)\|_{L^p}$.

Remark 2.1.4. One could replace the compact support assumptions by some logarithmic decrease of the solutions ω and ρ at infinity as done in [26] and [75] but for the sake of simplicity we will only consider solutions with compact support.

In order to show that the limit of a sequence (ω_N, Q_N, P_N) of solutions of (2.1.1) converges to a solution (ω, ρ, v) of (2.1.2), we will use the modulated energy introduced in Subsection 1.3.1.

Let $v \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$, ρ, ω, ω_N be $(L^1 \cap L^\infty)(\mathbb{R}^2, \mathbb{R})$ with compact supports and $Q_N, P_N \in (\mathbb{R}^2)^N$ be such that $q_i \neq q_j$ if $i \neq j$. Assume that

$$\int_{\mathbb{R}^2} \omega = \int_{\mathbb{R}^2} \omega_N$$
$$\int_{\mathbb{R}^2} \rho = 1.$$

Then this modulated energy is defined by:

$$\begin{aligned}
&\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N) \\
&\quad := \frac{1}{N} \sum_{i=1}^N |v(q_i) - p_i|^2 \\
&\qquad + \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta} g(x-y)(\rho + \omega - \rho_N - \omega_N)^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y) \\
&\qquad + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x-y)(\omega - \omega_N)(x)(\omega - \omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \|\omega - \omega_N\|_{L^2}^2 + BN^{-\gamma}
\end{aligned}$$

where

$$\rho_N := \frac{1}{N} \sum_{i=1}^N \delta_{q_i}$$

and γ and B are constants ensuring that $\mathcal{H}$ is nonnegative, as explained by the following result:

Proposition 2.1.5. *For any $0 < \gamma < 1$, there exists a constant B depending only on γ, $\|\omega\|_{L^1\cap L^\infty}$, $\|\rho\|_{L^1\cap L^\infty}$ and $\sup_N \|\omega_N\|_{L^1\cap L^\infty}$ such that:*

$$\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho+\omega-\rho_N-\omega_N)^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y) + BN^{-\gamma} \geqslant 0. \tag{2.1.4}$$

and

$$\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N) \geqslant 0.$$

This Proposition is similar to Proposition 1.2.3 (proved in [81]) which states that for a bounded probability density μ, we have

$$\mathcal{F}(X_N,\mu) \geqslant -\frac{\ln(N)}{2N} - CN^{-1}$$

where C depends only on $\|\mu\|_{L^\infty}$. Since we have no information on the sign of $\rho+\omega-\omega_N$, it is not necessarily a probability density. Therefore we will prove Proposition 2.1.5 in Section 2.3 using the method that was used to prove Proposition 1.2.3 in [81].

Remark that if we remove $BN^{-\gamma}$ and if we set $\omega_N = \omega = 0$ our quantity $\mathcal{H}$ is the functionnal $\mathcal{H}^c$ used by Duerinckx and Serfaty in the appendix of [81] to prove the mean-field limit of particles satisfying (1.2.9) to the Euler-Poisson equations (see the end of Subsection 1.2.2).

Our main result is the following theorem:

Theorem 2.1.6. *Let (ρ,ω,v) be a weak solution of System (2.1.2) in the sense of Definition 2.1.1 and (ω_N,Q_N,P_N) be a weak solution of System (2.1.1) in the sense of Definition 2.1.2 such that*

$$\int_{\mathbb{R}^2}\omega_{N,0} = \int_{\mathbb{R}^2}\omega_0. \tag{2.1.5}$$

Then we define

$$\mathcal{H}_N(t) := \mathcal{H}(\omega(t),\rho(t),v(t),\omega_N(t),Q_N(t),P_N(t)). \tag{2.1.6}$$

Suppose that $\nabla\omega \in L^\infty$, $\nabla v \in \mathcal{C}^0([0,T]\times\mathbb{R}^2,\mathbb{R}^2)$ and that

$$\sup_{N\in\mathbb{N}} \|\omega_{N,0}\|_{L^1\cap L^\infty} < +\infty \tag{2.1.7}$$

$$q_1(0),...,q_N(0) \notin \operatorname{supp}(\omega_{N,0}) \tag{2.1.8}$$

$$\forall i\neq j, q_i(0)\neq q_j(0). \tag{2.1.9}$$

Then there exist positive constants C and β depending only on T,ρ,ω and $\sup_{N\in\mathbb{N}}\|\omega_{N,0}\|_{L^1\cap L^\infty}$ such that for all $t\in[0,T]$,

$$\mathcal{H}_N(t) \leqslant C(\mathcal{H}_N(0)+N^{-\beta}). \tag{2.1.10}$$

Remark 2.1.7. By conservation of mass (2.1.5) implies that for all $t \in [0, T]$,

$$\int_{\mathbb{R}^2} \omega_N(t) = \int_{\mathbb{R}^2} \omega(t).$$

Remark 2.1.8. By Sobolev embeddings the solutions of the spray system (2.1.2) given by Theorem 2.2.1 are also solutions in the sense of Definition 2.1.1 that satisfy the hypothesis of Theorem 2.1.6 and thus Theorem 2.2.1 gives the existence of sufficiently regular solutions of System (2.1.2) that can be approached as mean-field limits of solutions of System (2.1.1) (even if Theorem 2.1.6 does not require solutions to be as regular as the solutions obtained in Theorem 2.2.1).

We will also prove a coerciveness result about this energy.

Proposition 2.1.9. *Let $Q_N, P_N \in (\mathbb{R}^2)^N$ and let $\omega, \omega_N, \rho \in L^1 \cap L^\infty(\mathbb{R}^2, \mathbb{R})$ with compact supports and $v \in W^{2,\infty}(\mathbb{R}^2, \mathbb{R}^2)$. Assume that*

$$\int_{\mathbb{R}^2} \omega = \int_{\mathbb{R}^2} \omega_N,$$
$$\int_{\mathbb{R}^2} \rho = 1$$

and that

$$\sup_{N \in \mathbb{N}} \|\omega_N\|_{L^1 \cap L^\infty} < +\infty. \tag{2.1.11}$$

Then there exist positive constants C and β such that

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \delta_{(q_i, p_i)} - \rho \otimes \delta_{\xi = v(x)} \right\|_{H^{-5}} \leqslant C(1 + \|\nabla v\|_{W^{1,\infty}}^2) \times (\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N)^{\frac{1}{2}} + (1 + \|\rho\|_{L^\infty}) N^{-\beta})^{\frac{1}{2}}. \tag{2.1.12}$$

In particular, if we assume that

$$\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N) \underset{N \to \infty}{\longrightarrow} 0$$

then for any $a < -1$,

$$\rho_N \underset{N \to +\infty}{\longrightarrow} \rho \quad \text{in } H^a$$
$$\omega_N - \omega \underset{N \to +\infty}{\longrightarrow} 0 \quad \text{in } L^2 \cap \dot{H}^{-1}$$
$$\frac{1}{N} \sum_{i=1}^{N} \delta_{(q_i, p_i)} \underset{N \to +\infty}{\longrightarrow} \rho \otimes \delta_{\xi = v(x)} \quad \text{in } H^{-5}.$$

Remark 2.1.10. The H^{-5} norm is not optimal, but it is sufficient to justify that $\mathcal{H}_N$ controls the convergence to a monokinetic distribution in a weak sense.

As a consequence we get that if a sequence of solutions (ω_N, Q_N, P_N) of (2.1.1) satisfying the hypothesis of Theorem 2.1.6 are such that

$$\mathcal{H}_N(0) \underset{N\to+\infty}{\longrightarrow} 0$$

then for any $t \in [0,T]$ we have

$$\mathcal{H}_N(t) \underset{N\to+\infty}{\longrightarrow} 0$$

and it follows by Proposition 2.1.9 that for any $t \in [0,T]$ and $a < -1$,

$$\begin{aligned} \rho_N(t) &\underset{N\to+\infty}{\longrightarrow} \rho(t) \;\text{ in } H^a \\ \omega_N(t) - \omega(t) &\underset{N\to+\infty}{\longrightarrow} 0 \;\text{ in } L^2 \cap \dot{H}^{-1} \\ \frac{1}{N}\sum_{i=1}^N \delta_{(q_i(t),p_i(t))} &\underset{N\to+\infty}{\longrightarrow} \rho(t) \otimes \delta_{\xi=v(t,x)} \;\text{ in } H^{-5}. \end{aligned}$$

Since $\frac{1}{N}\sum_{i=1}^N \delta_{(q_i(t),p_i(t))}$ is bounded in the dual of continuous bounded functions, we can extract a subsequence which will converge for the weak-$*$ topology of probability measures. Since it necessarily converges to $\rho(t) \otimes \delta_{\xi=v(t,x)}$, by weak-$*$ compactness we can deduce that for all $t \in [0,T]$,

$$\frac{1}{N}\sum_{i=1}^N \delta_{(q_i(t),p_i(t))} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \rho(t) \otimes \delta_{\xi=v(t,x)}$$

for the weak-$*$ topology of probability measures and thus we have the mean-field convergence of (2.1.1) to (2.1.2). One can look at [75, Corollary 1.2] for a more detailed proof of such a compactness argument.

Remark 2.1.11. If we suppose that $\omega_{N,0} - \omega_0$ converges to 0 in $\dot{H}^{-1}$ and that $\rho_{N,0}$ converges to ρ_0 in the weak-$*$ topology of signed measures, then the convergence of

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho_0 + \omega_0 - \rho_{N,0} - \omega_{N,0})^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)$$

to zero can be seen as a well-preparedness condition on the initial data, as stated in the following proposition:

Proposition 2.1.12. *Let us suppose that $\omega_{N,0}, \omega_0, \rho_0 \in L^2(\mathbb{R}^2, \mathbb{R})$ with compact support and that $(q_1^0, ..., q_N^0)$ are such that $q_i^0 \neq q_j^0$ if $i \neq j$. Then if we suppose*

$$\int_{\mathbb{R}^2} \omega_0 = \int_{\mathbb{R}^2} \omega_{N,0},$$
$$\int_{\mathbb{R}^2} \rho_0 = 1,$$

$$\sup_{N\in\mathbb{N}} \|\omega_{N,0}\|_{L^1\cap L^\infty} < +\infty \tag{2.1.13}$$

and

$$\omega_{N,0} - \omega_0 \underset{N\to+\infty}{\longrightarrow} 0 \ \text{ in } \dot{H}^{-1}$$
$$\rho_{N,0} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \rho_0 \ \text{ in } \mathcal{P}(\mathbb{R}^2)$$
$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g(q_i^0 - q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\rho_0(x)\rho_0(y)\,\mathrm{d}x\,\mathrm{d}y$$

we have

$$\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\rho_0 + \omega_0 - \rho_{N,0} - \omega_{N,0})^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y) \underset{N\to+\infty}{\longrightarrow} 0.$$

The latter statement strongly relies on Proposition 1.2.2 (proved in [26]). One could have more details about these well-preparedness assumptions by reading the introduction of [26].

Let us finish this introduction by giving some functionnal inequalities on the Coulomb modulated energy introduced by Duerinckx and Serfaty in [26, 81] (see Subsection 1.2.2) that we will need in Section 2.3. For $X_N = (x_1, ..., x_N) \in (\mathbb{R}^2)^N$ such that $x_i \neq x_j$ if $i \neq j$ and μ a L^∞ probability density with compact support, this energy is defined by:

$$\mathcal{F}(X_N, \mu) := \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\left(\mu - \sum_{i=1}^N \delta_{x_i}\right)(\mathrm{d}x)\left(\mu - \sum_{i=1}^N \delta_{x_i}\right)(\mathrm{d}y). \tag{2.1.14}$$

As we explained in Subsection 1.2.2 this quantity controls the distance between μ and the empirical distribution on X_N in a weak sense. More precisely we have the following proposition proved in [81] (number 3.6 in the article):

Proposition 2.1.13 (proved in [81]). *For any $0 < \theta \leqslant 1$, there exists $\lambda > 0$ and $C > 0$ such that for ξ smooth and $\mu \in L^\infty$ probability density with compact support,*

$$\left|\int_{\mathbb{R}^2} \xi \left(\mu - \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}\right)\right| \leqslant C\Bigg(|\xi|_{\mathcal{C}^{0,\theta}} N^{-\lambda} \\ + \|\nabla \xi\|_{L^2} \left(\mathcal{F}(X_N,\mu) + (1+\|\mu\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\right)^{\frac{1}{2}}\Bigg)$$

where

$$|\xi|_{\mathcal{C}^{0,\theta}} := \sup_{x\neq y} \frac{|\xi(x)-\xi(y)|}{|x-y|^\theta}.$$

Remark 2.1.14. In [81, Proposition 3.6] the coercivity inequality is stated with the Hölder norm $\|\xi\|_{\mathcal{C}^{0,\theta}}$ but by inequality [81, Inequality (3.27)] we can replace this Hölder norm by the semi-norm $|\xi|_{\mathcal{C}^{0,\theta}}$.

We will also need the following functional inequality, proved by Serfaty in [81] (number 1.1 in the article).

Proposition 2.1.15. *There exists $\lambda, C > 0$ such that for any probability density $\mu \in L^\infty$ with compact support, $\psi \in W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)$ and $X_N \in (\mathbb{R}^2)^N$, we have*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (\psi(x)-\psi(y))\cdot\nabla g(x-y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}-\mu\right)(x)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}-\mu\right)(y) \\ \leqslant C\,\|\psi\|_{W^{1,\infty}}\left(\mathcal{F}(X_N,\mu) + (1+\|\mu\|_{L^\infty})N^{-\lambda}\right).$$

This proposition is one of the main result of [81] as it is used to perform a Grönwall estimate on the modulated energy from which the mean-field result is deduced.

The remainder of this paper is organized as follows. In Section 2.2 we establish local well-posedness of strong solutions of (2.1.2). Then in Section 2.3 we provide the proof of Proposition 2.1.5, Theorem 2.1.6, Proposition 2.1.9 and Proposition 2.1.12. Sections 2.2 and 2.3 are independent of each other.

2.2 Local Well-Posedness

In this section, if μ is a continuous function defined on $\mathbb{R}^2$ with compact support, we will denote

$$R[\mu] := \sup\,\{|x|\ ;\ x\in\mathbb{R}^2, \mu(x)\neq 0\}$$

and

$$R_T[\mu] := \sup_{0\leqslant t\leqslant T} R[\mu(t)]$$

if μ depends on time. If B is a Banach space and $1 \leqslant p \leqslant \infty$, we will denote

$$L^p_T B := L^p([0,T],B).$$

We will use the same convention for the spaces $\mathcal{C}_T^k B$ and the Sobolev spaces $W_T^{k,p} B$. Let us also recall that $g(x) = -\frac{1}{2\pi} \ln|x|$ is the opposite of the Green kernel on the plane.

C will refer to a constant independent of time and of any other parameter that can change value from one line to another. We will denote $C(A, B)$ for a constant depending only on some quantities A and B.

We want to show that System (2.1.2) has a unique regular solution on $[0, T]$ for T small enough. In [62], Makino builds such a solution for the following compressible Euler-Poisson system in three dimensions:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = F * \rho \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \end{cases}$$

where p is a function of ρ and $F := \nabla G$ where G is the Green function on $\mathbb{R}^3$. There are three main differences with our system (2.1.2):

1. We have no pressure term, but we have a gyroscopic effect.
2. We have a continuity equation on ω that we also need to solve.
3. On the plane $\mathbb{R}^2$, the function $V = -\nabla^\perp g * (\rho + \omega)$ is not in L^2 except if we assume that $\int (\rho + \omega) = 0$.

In order to deal with the third point, we will assume that $v_0 = u_0 + \overline{V}$ where $u_0 \in L^2$ and $\overline{V}$ is a function of x that we will specify later. If we try to find a solution of (2.1.2) when $v = u + \overline{V}$, we find that (u, ρ, ω) evolves according to the following equations:

$$\begin{cases} \partial_t \omega + \operatorname{div}(\omega V) = 0 \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t u + ((u + \overline{V}) \cdot \nabla) u + (u \cdot \nabla) \overline{V} = u^\perp + f \\ V = -\nabla^\perp g * (\omega + \rho) \\ f = (\overline{V} - V)^\perp - (\overline{V} \cdot \nabla) \overline{V} \\ v = u + \overline{V}. \end{cases} \tag{2.2.1}$$

Thus if we choose $\overline{V}$ such that $f \in L^2$, we will find an equation that we expect to have a solution in L^2. We can achieve this goal choosing the following value of $\overline{V}$:

$$\overline{V} := -\left(\int_{\mathbb{R}^2} \omega_0 + \rho_0 \right) \nabla^\perp g * \chi$$

where χ is some compactly supported function such that $\int_{\mathbb{R}^2} \chi = 1$. We make such a choice because $\int_{\mathbb{R}^2} \rho$ and $\int_{\mathbb{R}^2} \omega$ are conserved and we will justify later that for μ compactly supported,

$$-\nabla^\perp g * \mu(x) = \frac{1}{2\pi}\left(\int_{\mathbb{R}^2}\mu\right)\frac{x^\perp}{|x|^2} + \underset{|x|\to\infty}{\mathcal{O}}(|x|^{-2}).$$

Since we assumed that ρ and ω have compact support, we are not concerned by the fact that V is not L^2 on the whole plane. Remark also that the space

$$-\left(\int_{\mathbb{R}^2}\omega_0+\rho_0\right)\nabla^\perp g * \chi + L^2$$

does not depend on the choice of χ. Now we are able to write the main theorem of this section:

Theorem 2.2.1. *Let s be an integer such that $s \geqslant 3$ and $u_0 \in H^{s+1}(\mathbb{R}^2,\mathbb{R}^2)$, $\rho_0,\omega_0 \in H^s(\mathbb{R}^2,\mathbb{R})$ such that ω_0 and ρ_0 have compact support, then if T is small enough (with respect to some quantity depending only on $\|\omega_0\|_{H^s}$, $\|\rho_0\|_{H^s}$, $\|u_0\|_{H^{s+1}}$, $R[\omega_0]$ and $R[\rho_0]$), there exists a unique (u,ω,ρ) with $(\rho,\omega)\in(\mathcal{C}^0_TH^s\cap\mathcal{C}^1_TH^{s-1})^2$ and $u\in\mathcal{C}^0_TH^{s+1}\cap\mathcal{C}^1_TH^s$ solution of (2.2.1).*

The proof of Theorem 2.2.1 proceeds as follows:

1. We fix $T>0$ and define

$$R_0 := R[\rho_0+\omega_0]$$
$$M_0 := \max(\|\rho_0\|_{H^s}, \|\omega_0\|_{H^s}, \|u_0\|_{H^{s+1}})$$

(2.2.2)
$$\begin{aligned} X_T := \Big\{(\omega,\rho)\in L^\infty_TH^s\cap\mathcal{C}^0_TH^{s-1}\Big|\,&\omega(0)=\omega_0,\rho(0)=\rho_0,\\ &\|\rho\|_{L^\infty_TH^s}\leqslant 2M_0, \|\omega\|_{L^\infty_TH^s}\leqslant 2M_0, R_T[\rho+\omega]\leqslant 2R_0,\\ &\forall t\in[0,T], \int(\rho(t)+\omega(t)) = \int(\rho_0+\omega_0),\\ &\forall t,t'\in[0,T], \left\|\rho(t)-\rho(t')\right\|_{H^{s-1}}\leqslant L|t-t'|,\\ &\left\|\omega(t)-\omega(t')\right\|_{H^{s-1}}\leqslant L|t-t'|,\Big\}\end{aligned}$$

where $L>0$ is a quantity depending only on R_0 and M_0. Remark that X_T is a subspace of $(\mathcal{C}^0_TH^s\cap\mathcal{C}^1_TH^{s-1})^2$. Then we fix $(\omega,\rho)\in X_T$ and we define

$$V := -\nabla^\perp g * (\rho+\omega) \tag{2.2.3}$$

$$\overline{V} := -\left(\int_{\mathbb{R}^2}\omega_0+\rho_0\right)\nabla^\perp g*\chi \tag{2.2.4}$$

$$f := (\overline{V}-V)^\perp - \overline{V}\cdot\nabla\overline{V}. \tag{2.2.5}$$

Note that we will prove in Subsection 2.2.1 that $f\in\mathcal{C}^0_TH^s\cap L^\infty_TH^{s+1}$.

2. In Subsection 2.2.2 we solve in $\mathcal{C}^0_{T_1}H^{s+1} \cap \mathcal{C}^1_{T_1}H^s$ the equation

$$\partial_t u + ((u + \overline{V}) \cdot \nabla)u + (u \cdot \nabla)\overline{V} = u^\perp + f$$

with initial condition u_0 and $T_1 \leqslant T$ small enough depending only on M_0 and L.

3. In Subsection 2.2.3 we define $v = u+\overline{V}$ and solve in $(\mathcal{C}_{T_1}H^s \cap \mathcal{C}^1_{T_1}H^{s-1})^2$ the system

$$\begin{cases} \partial_t \widetilde{\omega} + \operatorname{div}(\widetilde{\omega}V) = 0 \\ \partial_t \widetilde{\rho} + \operatorname{div}(\widetilde{\rho}v) = 0. \end{cases}$$

4. In Subsection 2.2.4 we apply a fixed-point theorem by showing that the map defined on X_{T_2} by $(\omega, \rho) \longmapsto (\widetilde{\omega}, \widetilde{\rho})$ is a contraction for the $\mathcal{C}^0_T L^2$ norm if $T_2 \leqslant T_1$ is small enough, using the estimates proved for the previous equations.

Remark 2.2.2. X_T is strictly included in $(\mathcal{C}^0_{T_1}H^s \cap \mathcal{C}^1_{T_1}H^{s-1})^2$ but since we prove in step (3) that the image of the application Φ sending (ω, ρ) to $(\widetilde{\omega}, \widetilde{\rho})$ is contained in $(\mathcal{C}^0_{T_1}H^s \cap \mathcal{C}^1_{T_1}H^{s-1})^2$ we have the expected regularity for the solutions of our system.

Remark 2.2.3. Uniqueness is established in the space X_T which is bigger than $(\mathcal{C}^0_T H^s \cap \mathcal{C}^1_T H^{s-1})^2$. It ensures uniqueness for the whole system: If (ρ_1, ω_1, u_1) and (ρ_2, ω_2, u_2) are two solutions of (2.1.2), then $(\rho_1, \omega_1) = (\rho_2, \omega_2)$ by uniqueness of the fixed point and $u_1 = u_2$ follows by uniqueness of solutions of Equation (2.2.6). Remark also that by using energy estimates one could prove uniqueness in a space of smaller regularity.

Before doing these different steps, we give some results about the Biot-Savart kernel $-\nabla^\perp g$ that we will need later. In this section we will use the following definition of uniformly local Sobolev spaces:

Definition 2.2.4. *We define $H^s_{\mathrm{ul}}(\mathbb{R}^2)$ as the space of locally H^s functions verifying*

$$\|u\|_{H^s_{\mathrm{ul}}(\mathbb{R}^2)} := \sup_{x \in \mathbb{R}^2} \|u\|_{H^s(B(x,1))} < +\infty.$$

For a more complete introduction to these spaces we refer to [50, Section 2.2].

2.2.1 Properties of the Biot-Savart kernel on the plane

In this subsection we prove Proposition 2.2.5, which contains several results about the Biot-Savart kernel $-\nabla^\perp g$.

Proposition 2.2.5. *Let $s \geqslant 3$ and let μ be a H^s function on $\mathbb{R}^2$ with compact support. Denote*

$$V := -\nabla^\perp g * \mu.$$

Then we have the following inequalities:

1. $V \in H^{s+1}_{\text{ul}}$ *and* $\|V\|_{H^{s+1}_{\text{ul}}} \leqslant C(1+R[\mu]) \|\mu\|_{H^s}$.
2. $\|\nabla V\|_{H^s} \leqslant C \|\mu\|_{H^s}$.
3. $V \in L^\infty$ *and we have the three following bounds:*

$$\|V\|_{L^\infty} \leqslant CR[\mu] \|\mu\|_{L^\infty}$$
$$\|V\|_{L^\infty} \leqslant CR[\mu]^{\frac{1}{3}} \|\mu\|_{H^1}$$
$$\|V\|_{L^\infty} \leqslant C \|\mu\|_{L^1}^{\frac{1}{2}} \|\mu\|_{L^\infty}^{\frac{1}{2}}.$$

4. $\|(V \cdot \nabla)V\|_{H^s} \leqslant C(1+R[\mu]) \|\mu\|_{H^s}^2$.
5. *If* μ *has average zero, then* $V \in L^2$ *and*

$$\|V\|_{L^2} \leqslant C(1+R[\mu]^2) \|\mu\|_{L^2}.$$

The fact that if μ has average zero and compact support then $V \in L^2$ is a classical result in fluid dynamics (see for example [61, Proposition 3.3]) that we will prove to have the specific bound we need on $\|V\|_{L^2}$. Estimates (1) to (4) are consequences of the two following propositions. The first one is the usual potential estimate of a velocity field given by the Biot-Savart law:

Proposition 2.2.6 (Potential estimates in L^p)**.** *If* $2 < p < \infty$ *and* $\omega \in L^1 \cap L^p$, *then*

$$\|\nabla g * \omega\|_{L^\infty} \leqslant C_p \|\omega\|_{L^1}^{\frac{p-2}{2p-2}} \|\omega\|_{L^p}^{\frac{p}{2p-2}}$$
$$\|\nabla g * \omega\|_{L^\infty} \leqslant C \|\omega\|_{L^1}^{\frac{1}{2}} \|\omega\|_{L^\infty}^{\frac{1}{2}}.$$

For the proof of this proposition see for example [46, Lemma 1]. The second is the Calderón-Zygmund inequality:

Proposition 2.2.7 (Calderón-Zygmund inequality)**.** *If* $1 < p < +\infty$,

$$\left\|\nabla^2 g * \omega\right\|_{L^p} \leqslant C_p \|\omega\|_{L^p}.$$

For the proof of this inequality we refer to [3, Proposition 7.5].

The first consequence of Proposition 2.2.5 is the following:

Corollary 2.2.8. *Let* $\rho_0, \omega_0 \in H^s$ *with compact support,* χ *be a smooth function with compact support and* $(\rho, \omega) \in X_T$ *where* X_T *is the space defined by* (2.2.2)*. Let us consider the functions* $\overline{V}$ *and* f *defined by* (2.2.4) *and* (2.2.5)*, then we have* $\overline{V} \in H^{s+2}_{\text{ul}}$, $f \in L^\infty_T H^{s+1} \cap \mathcal{C}^0_T H^s$ *and*

$$\left\|\overline{V}\right\|_{H^{s+2}_{\text{ul}}} \leqslant C(R_0, M_0)$$
$$\|f\|_{L^\infty_T H^{s+1}} \leqslant C(R_0, M_0).$$

Proof of Proposition 2.2.5. Let us begin by the second inequality. We have:

$$\|\nabla V\|_{H^s} = \left\|\nabla^2 g * \mu\right\|_{H^s} \leqslant C \sum_{|\alpha|\leqslant s} \left\|\nabla^2 g * \partial^\alpha \mu\right\|_{L^2} \leqslant C \|\mu\|_{H^s}$$

by Proposition 2.2.7.

Let us now prove the third Claim. By Proposition 2.2.6, we have

$$\begin{aligned}
\|V\|_{L^\infty} &\leqslant C \|\mu\|_{L^1}^{\frac{1}{2}} \|\mu\|_{L^\infty}^{\frac{1}{2}} \\
&\leqslant C \|\mu\|_{L^\infty}^{\frac{1}{2}} \left(\int_{B(0,R[\mu])} |\mu|\right)^{\frac{1}{2}} \\
&\leqslant C \|\mu\|_{L^\infty} \left(\int_{B(0,R[\mu])} 1\right)^{\frac{1}{2}} \\
&\leqslant CR[\mu] \|\mu\|_{L^\infty}.
\end{aligned}$$

For the second inequality of (3), we use Proposition 2.2.6 again to get

$$\|V\|_{L^\infty} \leqslant C \|\mu\|_{L^1}^{\frac{1}{3}} \|\mu\|_{L^4}^{\frac{2}{3}}.$$

Moreover, by Cauchy-Schwartz inequality,

$$\|\mu\|_{L^1} \leqslant C \left\|\mathbf{1}_{B(0,R[\mu])}\right\|_{L^2} \|\mu\|_{L^2} \leqslant CR[\mu] \|\mu\|_{H^1}.$$

and therefore by the embedding of H^1 into L^4 (see for example [14, Corollary 9.11]) we have

$$\|V\|_{L^\infty} \leqslant CR[\mu]^{\frac{1}{3}} \|\mu\|_{H^1}.$$

The third inequality of (3) is the second inequality of Proposition 2.2.6.

The first inequality follows from the two Claims we just proved: Since all derivatives of V of order k for $1 \leqslant k \leqslant s+1$ belong to L^2 and since $\|V\|_{L^2_{\mathrm{ul}}} \leqslant C \|V\|_{L^\infty}$, we get

$$\begin{aligned}
\|V\|_{H^{s+1}_{\mathrm{ul}}} &\leqslant C(\|\mu\|_{H^s} + R[\mu] \|\mu\|_{L^\infty}) \\
&\leqslant C(1 + R[\mu]) \|\mu\|_{H^s}
\end{aligned}$$

because $H^s \hookrightarrow L^\infty$.

Now let us prove the fourth point. Let α be a multi-index such that $|\alpha| \leqslant s$, then $\partial^\alpha((V\cdot\nabla)V)$ is a combination of $(\partial^{\alpha_1}V\cdot\nabla)\partial^{\alpha_2}V$ where $\alpha_1+\alpha_2=\alpha$. If $\alpha_1 = 0$,

$$\|(\partial^{\alpha_1}V\cdot\nabla)\partial^{\alpha_2}V\|_{L^2} \leqslant \|V\|_{L^\infty} \|\nabla V\|_{H^s}.$$

If $1 \leqslant |\alpha_1| \leqslant s-1$, then

$$\|(\partial^{\alpha_1} V \cdot \nabla)\partial^{\alpha_2} V\|_{L^2} \leqslant \|\partial^{\alpha_1} V\|_{L^\infty} \|\nabla V\|_{H^s} \leqslant \|\nabla V\|_{H^s}^2 .$$

Finally if $|\alpha_1| = s$,

$$\|(\partial^{\alpha_1} V \cdot \nabla)\partial^{\alpha_2} V\|_{L^2} \leqslant \|\partial^{\alpha_1} V\|_{L^2} \|\nabla V\|_{L^\infty} \leqslant \|\nabla V\|_{H^s}^2 .$$

We conclude using (2) and (3).

Now let us assume that $\int_{\mathbb{R}^2} \mu = 0$ and bound the L^2 norm of V. We have

$$\widehat{V}(\xi) = -\frac{\xi^\perp}{|\xi|^2}\widehat{\mu}(\xi)$$

and therefore by Fourier-Plancherel,

$$\|V\|_{L^2}^2 = C \int_{\mathbb{R}^2} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2} \,\mathrm{d}\xi = C \|\mu\|_{\dot{H}^{-1}} .$$

Now we write

$$\|\mu\|_{\dot{H}^{-1}}^2 = \int_{|\xi|\leqslant 1} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2} \,\mathrm{d}\xi + \int_{|\xi|\geqslant 1} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2} \,\mathrm{d}\xi.$$

By Fourier-Plancherel,

$$\int_{|\xi|\geqslant 1} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2} \,\mathrm{d}\xi \leqslant \int_{\mathbb{R}^2} |\widehat{\mu}(\xi)|^2 \,\mathrm{d}\xi \leqslant C \|\mu\|_{L^2}^2 .$$

Since μ has average zero, for $|\xi| \leqslant 1$,

$$\begin{aligned}
|\widehat{\mu}(\xi)| &= \left| \int_{\mathbb{R}^2} \mu(x)(e^{-ix\cdot\xi} - 1) \,\mathrm{d}x \right| \\
&\leqslant C \int_{\mathbb{R}^2} |\mu(x)| \left| \sin\left(\frac{x\cdot\xi}{2}\right) \right| \,\mathrm{d}x \\
&\leqslant C|\xi| \int_{\mathbb{R}^2} |x||\mu(x)| \,\mathrm{d}x \\
&\leqslant C|\xi| \left(\int_{B(0,R[\mu])} |x|^2 \right)^{\frac{1}{2}} \|\mu\|_{L^2} \\
&\leqslant CR[\mu]^2 \|\mu\|_{L^2} .
\end{aligned}$$

Therefore

$$\int_{|\xi|\leqslant 1} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2} \,\mathrm{d}\xi \leqslant CR[\mu]^4 \|\mu\|_{L^2}^2$$

so we get the proof of Claim (5). □

Now we prove the uniform bounds we need on f and $\overline{V}$:

Proof of Corollary 2.2.8. First remark that

$$\begin{aligned}
\|\overline{V}\|_{H^{s+2}_{\mathrm{ul}}} &= \left|\int_{\mathbb{R}^2} \rho_0 + \omega_0\right| \|\nabla g * \chi\|_{H^{s+2}_{\mathrm{ul}}} \\
&\leqslant C \left|\int \rho_0 + \omega_0\right| \\
&\leqslant C \|\rho_0 + \omega_0\|_{L^1} \\
&\leqslant CR[\rho_0 + \omega_0] \|\rho_0 + \omega_0\|_{L^2} \\
&\leqslant 2CR_0 M_0
\end{aligned}$$

by Claims (1) and (3) of Proposition 2.2.5. Moreover, if we denote

$$h = \rho + \omega - \left(\int_{\mathbb{R}^2} \rho_0 + \omega_0\right) \chi$$

we have

$$\begin{aligned}
\|f\|_{L^\infty_T H^{s+1}} &\leqslant \left\|(V - \overline{V})^\perp - (\overline{V} \cdot \nabla)\overline{V}\right\|_{L^\infty_T H^{s+1}} \\
&\leqslant \left\|(V - \overline{V})^\perp\right\|_{L^\infty_T L^2} + \left\|\nabla (V - \overline{V})^\perp\right\|_{L^\infty_T H^s} \\
&\quad + \left\|(\overline{V} \cdot \nabla)\overline{V}\right\|_{L^\infty_T H^{s+1}} \\
&\leqslant \|\nabla g * h\|_{L^\infty_T L^2} + \left\|\nabla^2 g * h\right\|_{L^\infty_T H^s} + \left\|(\overline{V} \cdot \nabla)\overline{V}\right\|_{L^\infty_T H^{s+1}} \\
&\leqslant C(1 + R_T[h]^2) \|h\|_{L^\infty_T L^2} + C \|h\|_{L^\infty_T H^s} \\
&\quad + C(1 + R_T[\chi]) \left|\int \rho_0 + \omega_0\right|^2 \|\chi\|^2_{L^\infty_T H^{s+1}} \\
&\leqslant C(R_0, M_0)
\end{aligned}$$

where we used Claims (2), (4) and (5) of Proposition 2.2.5.

Now let us justifiy that $f \in \mathcal{C}^0_T H^s$. If $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned}
\|f(t_1) - f(t_2)\|_{H^s} &= \left\|\nabla^\perp g * (\rho(t_1) + \omega(t_1) - \rho(t_2) - \omega(t_2))\right\|_{H^s} \\
&\leqslant \left\|\nabla^\perp g * (\rho(t_1) + \omega(t_1) - \rho(t_2) - \omega(t_2))\right\|_{L^2} \\
&\quad + \left\|\nabla^2 g * (\rho(t_1) + \omega(t_1) - \rho(t_2) - \omega(t_2))\right\|_{H^{s-1}} \\
&\leqslant C(1 + R_T[\rho + \omega]^2) \\
&\quad \times \|\rho(t_1) + \omega(t_1) - \rho(t_2) - \omega(t_2)\|_{L^2} \\
&\quad + C \|\rho(t_1) + \omega(t_1) - \rho(t_2) - \omega(t_2)\|_{H^{s-1}}
\end{aligned}$$

where we used points (5) and (2) of Proposition 2.2.5 and therefore $f \in \mathcal{C}^0_T H^s$ follows from $\rho, \omega \in \mathcal{C}^0_T H^{s-1}$. ☐

2.2.2 Pressureless Euler equations

In this subsection we prove that there is a unique solution to the following equation

$$\partial_t u + ((u + \overline{V}) \cdot \nabla)u + (u \cdot \nabla)\overline{V} = u^\perp + f \tag{2.2.6}$$

where $\overline{V}$ and f are the functions defined in (2.2.4) and (2.2.5).

Following the idea of [50, 62], we start by fixing $u \in \mathcal{C}^0_T L^2 \cap L^\infty_T H^{s+1}$ and we solve the linearized equation:

$$\begin{cases} \partial_t \widetilde{u} + ((u + \overline{V}) \cdot \nabla)\widetilde{u} + (\widetilde{u} \cdot \nabla)\overline{V} = \widetilde{u}^\perp + \widetilde{f} \\ \widetilde{u}(0) = \widetilde{u}_0. \end{cases} \tag{2.2.7}$$

We have the following well-posedness theorem:

Theorem 2.2.9. *If s is an integer such that $s \geqslant 3$, $u \in \mathcal{C}^0_T L^2 \cap L^\infty_T H^{s+1}$, $\widetilde{u}_0 \in H^{s+1}$, μ with compact support and $\widetilde{f} \in L^1_T H^{s+1} \cap \mathcal{C}^0_T H^s$, then (2.2.7) has a solution $\widetilde{u} \in \mathcal{C}^0_T H^{s+1} \cap \mathcal{C}^1_T H^s$, unique in the space $\mathcal{C}^0_T H^1 \cap \mathcal{C}^1_T L^2$. Moreover, we have the following estimates:*

$$\|\widetilde{u}(t)\|_{H^{s+1}} \leqslant e^{CT(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}} + 1)} \left(\|\widetilde{u}_0\|_{H^{s+1}} + C \left\| \widetilde{f} \right\|_{L^1_T H^{s+1}} \right)$$

$$\|\partial_t \widetilde{u}(t)\|_{H^s} \leqslant C \left(\left\| \widetilde{f}(t) \right\|_{H^s} + (\|u\|_{L^\infty_T H^{s+1}} + \left\|\overline{V}\right\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + 1) \|\widetilde{u}(t)\|_{H^{s+1}} \right).$$

Proof. The proof is a direct application of Theorem 1 of [50] which gives the well-posedness result and the estimates: We can rewrite (2.2.7) as

$$\partial_t \widetilde{u} + \sum_{i=1}^{2} A_i \partial_i \widetilde{u} + A_3 \widetilde{u} = f$$

where $A_i := (u_i + \overline{V}_i) I_2$ for $i \in \{1, 2\}$ and $A_3 := \begin{pmatrix} \partial_1 \overline{V}_1 & \partial_2 \overline{V}_1 + 1 \\ \partial_1 \overline{V}_2 - 1 & \partial_2 \overline{V}_2 \end{pmatrix}$.

To apply the theorem we need to prove the following:

1. $A_i \in \mathcal{C}^0_T L^2_{\mathrm{ul}}$ for $1 \leqslant i \leqslant 3$
2. $\forall t \in [0, T]\ \|A_i(t)\|_{s+1,\mathrm{ul}} \leqslant K$ for $1 \leqslant i \leqslant 3$
3. A_1 and A_2 symmetric
4. $\widetilde{f} \in L^1_T H^{s+1} \cap \mathcal{C}^0_T H^s$
5. $\widetilde{u}_0 \in H^{s+1}$

where $K := \|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}_{\mathrm{ul}}} + C$. The three last points are automatically checked by the assumptions of the theorem. For the first point and the second point, since $u \in \mathcal{C}^0_T L^2 \cap L^\infty_T H^{s+1}$, we only need to prove that $\overline{V}$ is in H^{s+2}_{ul}, which is given by Corollary 2.2.8.

□

As in [50] and [62] we will use the previous estimates to apply a fixed point theorem $u \mapsto \widetilde{u}$ on Equation (2.2.7) to prove the well-posedness of the non-linear equation (2.2.6). Let us first recall that we have fixed $u_0 \in H^{s+1}$, $(\omega, \rho) \in X_T$ (where X_T is defined by (2.2.2)) and

$$
\begin{aligned}
R_0 &:= R[\rho_0 + \omega_0] \\
M_0 &:= \max(\|\rho_0\|_{H^s}, \|\omega_0\|_{H^s}, \|u_0\|_{H^{s+1}}) \\
V &:= -\nabla^\perp g * (\rho + \omega) \\
\overline{V} &:= -\left(\int \omega_0 + \rho_0\right) \nabla^\perp g * \chi \\
f &:= (\overline{V} - V)^\perp - \overline{V} \cdot \nabla \overline{V}.
\end{aligned}
$$

Then the well-posedness of (2.2.6) is given by the following theorem:

Theorem 2.2.10. *Let s be an integer such that $s \geqslant 3$, then*

1. *There exists $T^* = T^*(M_0, R_0) \leqslant T$ such that if $T_1 \leqslant T^*$, there is a unique solution $u \in \mathcal{C}^0_{T_1} H^{s+1} \cap \mathcal{C}^1_{T_1} H^s$ to (2.2.6), with*

$$\|u\|_{L^\infty_{T_1} H^{s+1}} \leqslant 2M_0.$$

2. *Let u and u' be two solutions defined on $[0, T_1]$ with initial condition u_0 and forcing terms f and f', where*

$$
\begin{aligned}
f' &:= (\overline{V} - V')^\perp - \overline{V} \cdot \nabla \overline{V} \\
V' &:= -\nabla^\perp g * (\rho' + \omega')
\end{aligned}
$$

and $(\rho', \omega') \in X_T$. Then we have

$$\|u - u'\|_{L^\infty_{T_1} H^r} \leqslant C e^{C(M_0, R_0) T_1} \|V - V'\|_{L^1_{T_1} H^r}$$

where $0 \leqslant r \leqslant s$.

Proof. Let $T_1 \leqslant T$. We will use a fixed-point method on the following subset of $\mathcal{C}^0_{T_1} L^2$:

$$
\widetilde{X}_{T_1} := \left\{ u \in L^\infty_{T_1} H^{s+1} \cap \mathcal{C}^0_{T_1} H^s \,\middle|\, \|u\|_{L^\infty_{T_1} H^{s+1}} \leqslant 2M_0, u(0) = u_0, \right.
\left. \|u(t) - u(t')\|_{H^s} \leqslant \widetilde{L} |t - t'| \ \ \forall t, t' \in [0, T_1] \right\}
$$

where $\widetilde{L}$ is a constant which will be fixed later. Let $u \in \widetilde{X}_{T_1}$ and $\widetilde{u}$ be the solution of (2.2.7) associated to u. By Theorem 2.2.9, for $t \leqslant T_1$, we have:

$$\|\widetilde{u}(t)\|_{H^{s+1}} \leqslant e^{cT_1(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}} + 1)} \left(\|u_0\|_{H^s} + c \|f\|_{L^1_{T_1} H^{s+1}} \right)$$

$$\leqslant e^{cT_1(C(R_0,M_0)+2M_0+1)}(M_0 + cT_1 C(M_0,R_0))$$

by Corollary 2.2.8. Thus we get

$$\|\widetilde{u}(t)\|_{H^{s+1}} \leqslant 2M_0$$

if T_1 is small enough. Moreover, using Corollary 2.2.8 again, we get

$$\begin{aligned}\|\partial_t \widetilde{u}(t)\|_{H^s} &\leqslant c\left(\|f(t)\|_{H^s} + (\|u\|_{L^\infty_T H^{s+1}} + \|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + 1)\,\|\widetilde{u}\|_{H^{s+1}}\right)\\ &\leqslant c(C(M_0,R_0) + (2M_0 + C(M_0,R_0)+1)2M_0) =: \widetilde{L}\end{aligned}$$

Thus for all $T_1 \leqslant T^*$ we have built a map $\Psi : \widetilde{X}_{T_1} \longrightarrow \widetilde{X}_{T_1}$ such that $\Psi(u) = \widetilde{u}$, where $T^* = T^*(M_0,R_0)$. We will now show that Ψ is a contraction for the induced distance on X_{T_1}. Let u and w be two elements of X_{T_1} and set $U := u - w$. Then $\widetilde{U} := \widetilde{u} - \widetilde{w}$ satisfies:

$$\partial_t \widetilde{U} + ((u+\overline{V})\cdot\nabla)\widetilde{U} + (\widetilde{U}\cdot\nabla)\overline{V} = -(U\cdot\nabla)\widetilde{w} + \widetilde{U}^\perp.$$

Therefore since $(U\cdot\nabla)\tilde{w} \in \mathcal{C}^0_T L^2 \cap L^1_T H^1$ we can apply Theorem 1 from [50] to have the following estimate:

$$\begin{aligned}\left\|\widetilde{U}\right\|_{L^\infty_{T_1} L^2} &\leqslant e^{cT_1(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + \|u\|_{L^\infty_T H^{s+1}}+1)}\left(0 + c\,\|(U\cdot\nabla)\widetilde{w}\|_{L^1_{T_1} L^2}\right)\\ &\leqslant e^{cT_1(C(M_0,R_0)+2M_0+1)} cT_1 \|\nabla\widetilde{w}\|_{L^\infty_{T_1} L^\infty} \|U\|_{L^\infty_{T_1} L^2}\\ &\leqslant 4cM_0 T_1 e^{cT_1(C(M_0,R_0)+2M_0+1)} \|U\|_{L^\infty_{T_1} L^2}\end{aligned}$$

using Corollary 2.2.8 in the last inequality. Thus Ψ is a contraction if T is small enough, so since $\widetilde{X}_{T_1}$ is complete (this can be proved in the same way as the closedness of X_T which is proved in the beginning of section 2.2.4), it has a unique fixed point in $\widetilde{X}_{T_1}$, thus (2.2.6) has a unique solution for short time. Remark that the solution we find belongs to the space $\mathcal{C}^0_{T_1} H^{s+1} \cap \mathcal{C}^1_{T_1} H^s$ by Theorem 2.2.9.

Now let us prove the second point of our theorem: Let u and u' be two solutions associated to f_1 and f_2 defined on $[0,T_1]$ with $T_1 \leqslant T^*(M_0,R_0)$. Then $U := u - u'$ verifies:

$$\partial_t U + ((u+\overline{V})\cdot\nabla)U + (U\cdot\nabla)(\overline{V} + u') = U^\perp + F$$

where $F := f - f'$. We can rewrite this equation as

$$\partial_t U + \sum_{i=1}^{2} A_i \partial_i U + BU = F$$

where $A_i := (u_i + \overline{V}_i)I_2$ and $B := \begin{pmatrix} \partial_1 \overline{V}_1 + \partial_1 u_1' & 1 + \partial_2 \overline{V}_1 + \partial_2 u_1' \\ \partial_1 \overline{V}_2 + \partial_1 u_2' - 1 & \partial_2 \overline{V}_2 + \partial_2 u_2' \end{pmatrix}$.
Then by Theorem 1 of [50], for any $0 \leqslant r \leqslant s$ we have:

$$\begin{aligned} \|U\|_{L^\infty_{T_1} H^r} &\leqslant C e^{cT_1 (\|A_1\|_{L^\infty_{T_1} H^s_{\mathrm{ul}}} + \|A_2\|_{L^\infty_{T_1} H^s_{\mathrm{ul}}} + \|B\|_{L^\infty_{T_1} H^s_{\mathrm{ul}}})} \|F\|_{L^1_{T_1} H^r} \\ &\leqslant C e^{c(\|\overline{V}\|_{L^\infty_T H^{s+2}_{\mathrm{ul}}} + M_0 + 1)T_1} \|F\|_{L^1_{T_1} H^r} \\ &\leqslant C e^{c(C(M_0, R_0) + M_0 + 1)T_1} \|V - V'\|_{L^1_{T_1} H^r} \end{aligned}$$

where we used Corollary 2.2.8 in the last inequality. □

2.2.3 Continuity equations

In this subsection we still fix $s \geqslant 3$, $u \in \mathcal{C}^0_T H^{s+1} \cap \mathcal{C}^1_T H^s$, $(\rho, \omega) \in (\mathcal{C}^0_T H^s)^2$, $V := -\nabla^\perp g * (\rho + \omega)$, χ smooth with compact support such that $\int \chi = 1$, $\overline{V} := -\left(\int \omega_0 + \rho_0\right) \nabla^\perp g * \chi$, $v := u + \overline{V}$ and we consider the following continuity equations:

$$\begin{cases} \partial_t \widetilde{\omega} + \operatorname{div}(\widetilde{\omega} V) = 0 \\ \partial_t \widetilde{\rho} + \operatorname{div}(\widetilde{\rho} v) = 0 \end{cases} \tag{2.2.8}$$

with initial conditions (ρ_0, ω_0).

Theorem 2.2.11. *Let u,ρ,ω be as in the upper paragraph, there exists a solution $(\widetilde{\rho}, \widetilde{\omega}) \in \mathcal{C}^0_T H^s \cap \mathcal{C}^1_T H^{s-1}$ of (2.2.8), unique in $\mathcal{C}^0_T L^2$. Moreover, we have the following estimates:*

$$\begin{aligned} \|\widetilde{\rho}\|_{L^\infty_T H^s} &\leqslant \|\rho_0\|_{H^s} e^{cT\|u\|_{L^\infty_T H^s}} \exp\left(c e^{cT\|u\|_{L^\infty_T H^s}} T \|\nabla v\|_{L^\infty_T H^s}\right) \\ \|\widetilde{\omega}\|_{L^\infty_T H^s} &\leqslant \|\omega_0\| e^{cT\|\nabla V\|_{L^\infty_T H^s}} \\ \|\partial_t \widetilde{\omega}\|_{L^\infty_T H^{s-1}} &\leqslant C(1 + R_T[\rho + \omega]^{\frac{1}{3}}) \|\rho + \omega\|_{L^\infty_T H^s} \|\widetilde{\omega}\|_{L^\infty_T H^s} \\ \|\partial_t \widetilde{\rho}\|_{L^\infty_T H^{s-1}} &\leqslant C\left(\left|\int (\rho_0 + \omega_0)\right| + \|u\|_{L^\infty_T H^{s+1}}\right) \|\widetilde{\rho}\|_{L^\infty_T H^s}. \end{aligned}$$

Now let $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ be two solutions associated to two velocity fields $v_1 = u_1 + \overline{V}$ and $v_2 = u_2 + \overline{V}$ with same initial conditions, and $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ be two solutions associated to two velocity fields V_1 and V_2 with same initial conditions, then we have the following estimates:

$$\|\widetilde{\omega}_1 - \widetilde{\omega}_2\|_{L^\infty_T L^2} \leqslant CT \|V_1 - V_2\|_{L^\infty_T L^2} \|\widetilde{\omega}_2\|_{L^\infty_T H^3}$$

$$\|\widetilde{\rho}_1 - \widetilde{\rho}_2\|_{L^\infty_T L^2} \leqslant CT \|\widetilde{\rho}_2\|_{L^\infty_T H^3} \|v_2 - v_1\|_{L^\infty_T H^1} e^{cT\|u_1\|_{L^\infty_T H^3}}.$$

We will also give a general lemma to control the support of a compactly supported solution of a continuity equation:

Lemma 2.2.12. *If μ is the solution of the following continuity equation,*

$$\partial_t \mu + \mathrm{div}(\mu a) = 0$$

with $a \in \mathcal{C}^0_T W^{1,\infty}$ and μ_0 with compact support, then μ has compact support and

$$R_T[\mu] \leqslant R[\mu_0] + T \|a\|_{L^\infty_T L^\infty} . \tag{2.2.9}$$

In order to prove the main theorem we will need the following result:

Lemma 2.2.13. *If a is a Lipschitz vector field, $\mu_0 \in L^2$ and $f \in L^1_T L^2$ then there exists a unique solution of the continuity equation*

$$\partial_t \mu + \mathrm{div}(\mu a) = f$$

in $\mathcal{C}^0_T L^2$. Moreover we have the following estimate

$$\|\mu(t)\|_{L^2} \leqslant \left(\|\mu_0\|_{L^2} + \int_0^t \|f(\tau)\|_{L^2} \,\mathrm{d}\tau \right) e^{c\int_0^t \|\mathrm{div}(a)(\tau)\|_{L^\infty} \mathrm{d}\tau}. \tag{2.2.10}$$

Proof of Lemma 2.2.13. The existence and uniqueness of the solution in $\mathcal{C}^0_T L^2$ can be obtained by Theorem 3.19 and Remark 3.20 of [3]. Moreover by Proposition 6 of [2], we know that for all t and almost every x we have

$$\begin{aligned} \mu(t, X(t,x)) =& \mu_0(x) \\ &+ \int_0^t \Big(\mathrm{div}(a)(s, X(s,x)) \mu(s, X(s,x)) + f(s, X(s,x)) \Big) \,\mathrm{d}s \end{aligned}$$

where X is the flow associated to a. Let us denote $\overline{h}(t,x) = h(t, X(t,x))$ for any function h. Taking the L^2 norm of the upper inequality we get

$$\|\overline{\mu}(t)\|_{L^2} \leqslant \|\mu_0\|_{L^2} + \|\overline{f}\|_{L^1_T L^2} + \int_0^t \|\mathrm{div}(a)(s)\|_{L^\infty} \|\overline{\mu}(s)\|_{L^2} .$$

Thus by Grönwall lemma,

$$\|\overline{\mu}(t)\|_{L^2} \leqslant (\|\mu_0\|_{L^2} + \|\overline{f}\|_{L^1_T L^2}) e^{\|\mathrm{div}(a)\|_{L^1_T L^\infty}} .$$

Now remark that for any L^2 function g,

$$\int |g(X(t,x))|^2 \,\mathrm{d}x = \int |JX^t(x)| |g(x)|^2 \,\mathrm{d}x \qquad \leqslant \|g\|_{L^2}^2 \, e^{\|\mathrm{div}(a)\|_{L^1_T L^\infty}}$$

by of [2, Inequality (7)]. Using it for $\overline{\mu}$ and $\overline{f}$ we get inequality (2.2.10). □

Now we prove the main theorem of the section:

Proof of Theorem 2.2.11. Let us now use the previous lemma to prove the H^s bound on $\widetilde{\omega}$. Let α be a multi-index such that $|\alpha| \leqslant s$. Then, since V is divergent-free,

$$\partial_t \partial^\alpha \widetilde{\omega} + \operatorname{div}(V \partial^\alpha \widetilde{\omega}) = F^\alpha$$

where F^α is a combination of $\partial^{\alpha_1} V \cdot \partial^{\alpha_2} \nabla \omega$ with $|\alpha_1| + |\alpha_2| = s$, $|\alpha_2| \leqslant s-1$ and $|\alpha_1| \geqslant 1$. Thus by the upper estimate (2.2.10), since V is divergent-free, we have:

$$\|\partial^\alpha \widetilde{\omega}(t)\|_{L^2} \leqslant \left(\|\partial^\alpha \widetilde{\omega}_0\|_{L^2} + \int_0^t \|F^\alpha(\tau)\|_{L^2} \, \mathrm{d}\tau \right).$$

If $|\alpha_1| \leqslant s-1$, then

$$\begin{aligned}
\|\partial^{\alpha_1} V \cdot \partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} &\leqslant \|\partial^{\alpha_1} V\|_{L^\infty} \|\partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} \\
&\leqslant \|\partial^{\alpha_1} V\|_{H^2} \|\partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} \\
&\leqslant \|\nabla V\|_{H^s} \|\widetilde{\omega}\|_{H^s}.
\end{aligned}$$

If $|\alpha_1| = s$, then $\alpha_2 = 0$, thus

$$\begin{aligned}
\|\partial^{\alpha_1} V \cdot \partial^{\alpha_2} \nabla \widetilde{\omega}\|_{L^2} &\leqslant \|\nabla V\|_{H^s} \|\nabla \widetilde{\omega}\|_{L^\infty} \\
&\leqslant \|\nabla V\|_{H^s} \|\nabla \widetilde{\omega}\|_{H^2} \\
&\leqslant \|\nabla V\|_{H^s} \|\widetilde{\omega}\|_{H^s}.
\end{aligned}$$

Thus

$$\|\partial^\alpha \widetilde{\omega}(t)\|_{L^2} \leqslant \left(\|\partial^\alpha \widetilde{\omega}_0\|_{L^2} + c \int_0^t \|\nabla V(\tau)\|_{H^s} \|\widetilde{\omega}(\tau)\|_{H^s} \, \mathrm{d}\tau \right).$$

Summing over all indices α, we get

$$\|\widetilde{\omega}(t)\|_{H^s} \leqslant \left(\|\widetilde{\omega}_0\|_{H^s} + c \int_0^t \|\nabla V(\tau)\|_{H^s} \|\widetilde{\omega}(\tau)\|_{H^s} \, \mathrm{d}\tau \right).$$

By Grönwall's lemma we get the first inequality of our theorem. Now we will prove the estimate on $\widetilde{\rho}$. For a multi-index α with $|\alpha| \leqslant s$ we also have

$$\partial_t \partial^\alpha \widetilde{\rho} + \operatorname{div}(v \partial^\alpha \widetilde{\rho}) = F^\alpha.$$

Because v is not divergent-free, F^α is now a combination of $\partial^{\alpha_1} v \partial^{\alpha_2} \widetilde{\rho}$ where $|\alpha_1| + |\alpha_2| = s+1$, $|\alpha_1| \geqslant 1$ and $|\alpha_2| \leqslant s$. If $|\alpha_1| \leqslant s-1$, we have

$$\begin{aligned}
\|\partial^{\alpha_1} v \cdot \partial^{\alpha_2} \widetilde{\rho}\|_{L^2} &\leqslant \|\partial^{\alpha_1} v\|_{L^\infty} \|\partial^{\alpha_2} \widetilde{\rho}\|_{L^2} \\
&\leqslant \|\partial^{\alpha_1} v\|_{H^2} \|\partial^{\alpha_2} \widetilde{\rho}\|_{L^2} \\
&\leqslant \|\nabla v\|_{H^s} \|\widetilde{\rho}\|_{H^s}.
\end{aligned}$$

Now if $|\alpha_1| = s$ or $s+1$ (respectively $|\alpha_2| = 0$ or 1),

$$\begin{aligned}\|\partial^{\alpha_1} v \cdot \partial^{\alpha_2} \widetilde{\rho}\|_{L^2} &\leqslant \|\nabla v\|_{H^s} \|\partial^{\alpha_2} \widetilde{\rho}\|_{L^\infty} \\ &\leqslant \|\nabla v\|_{H^s} \|\partial^{\alpha_2} \widetilde{\rho}\|_{H^2} \\ &\leqslant \|\nabla v\|_{H^s} \|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

Thus

$$\begin{aligned}\|\partial^\alpha \widetilde{\rho}(t)\|_{L^2} \leqslant \Bigg(&\|\partial^\alpha \widetilde{\rho}_0\|_{L^2} \\ &+ c\int_0^t \|\nabla v(\tau)\|_{H^s} \|\widetilde{\rho}(\tau)\|_{H^s} \, \mathrm{d}\tau \Bigg) e^{c\int_0^T \|\mathrm{div}(v)\|_{L^\infty}(\tau)\,\mathrm{d}\tau}.\end{aligned}$$

Summing over all indices α, we get

$$\begin{aligned}\|\widetilde{\rho}(t)\|_{H^s} &\leqslant \left(\|\widetilde{\rho}_0\|_{H^s} + c\int_0^t \|\nabla v(\tau)\|_{H^s} \|\widetilde{\rho}(\tau)\|_{H^s} \, \mathrm{d}\tau \right) e^{c\int_0^T \|\mathrm{div}(v)\|_{L^\infty}(\tau)\,\mathrm{d}\tau} \\ &\leqslant \left(\|\widetilde{\rho}_0\|_{H^s} + c\int_0^t \|\nabla v(\tau)\|_{H^s} \|\widetilde{\rho}(\tau)\|_{H^s} \, \mathrm{d}\tau \right) e^{c\int_0^T \|u(\tau)\|_{H^s}\,\mathrm{d}\tau}\end{aligned}$$

because $\mathrm{div}(v) = \mathrm{div}(u)$. The corresponding estimate follows by Grönwall's lemma.

Now let us bound the time derivatives of $\widetilde{\omega}$ and $\widetilde{\rho}$. Take α a multi-index with $|\alpha| \leqslant s-1$, then

$$\partial_t \partial^\alpha \widetilde{\omega} = -\partial^\alpha (V \cdot \nabla \widetilde{\omega}) = \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2} \partial^{\alpha_1} V \cdot \nabla \partial^{\alpha_2} \widetilde{\omega}.$$

Moreover,

$$\begin{aligned}\|\partial^{\alpha_1} V \cdot \nabla \partial^{\alpha_2} \widetilde{\omega}\|_{L^2} &\leqslant \|\partial^{\alpha_1} V\|_{L^\infty} \|\nabla \partial^{\alpha_2} \widetilde{\omega}\|_{L^2} \\ &\leqslant C(\|V\|_{L^\infty} + \|\nabla V\|_{H^s}) \|\widetilde{\omega}\|_{H^s}.\end{aligned}$$

Now by Claim (3) of Proposition 2.2.5,

$$\|V\|_{L^\infty} \leqslant C R_T[\rho+\omega]^{\frac{1}{3}} \|\rho+\omega\|_{H^1}$$

and $\|\nabla V\|_{H^s} \leqslant C\|\rho+\omega\|_{H^s}$. Thus we have our estimate.

Let us do the same kind of computations for $\widetilde{\rho}$:

$$\partial_t \partial^\alpha \widetilde{\rho} = \partial^\alpha (\mathrm{div}(u)\widetilde{\rho} + u \cdot \nabla \widetilde{\rho} + \overline{V} \cdot \nabla \widetilde{\rho}).$$

If $|\alpha_1 + \alpha_2| = s-1$,

$$\begin{aligned}\|\partial^{\alpha_1} u \cdot \partial^{\alpha_2} \nabla \widetilde{\rho}\|_{L^2} &\leqslant \|\partial^{\alpha_1} u\|_{L^\infty} \left\|\partial^{\alpha_2} \widetilde{\nabla} \rho\right\|_{L^2} \\ &\leqslant \|u\|_{H^{s+1}} \|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

We do the same estimates for every term composing $\partial^\alpha(\operatorname{div}(u)\widetilde{\rho})$, except for

$$\begin{aligned}\|\partial^\alpha \operatorname{div}(u)\widetilde{\rho}\|_{L^2} &\leqslant \|u\|_{H^s}\|\widetilde{\rho}\|_{L^\infty}\\ &\leqslant \|u\|_{H^{s+1}}\|\widetilde{\rho}\|_{H^s}.\end{aligned}$$

Now for the third term, if $|\alpha_1+\alpha_2| = s-1$,

$$\begin{aligned}\left\|\partial^{\alpha_1}\overline{V}\cdot\nabla\partial^{\alpha_2}\widetilde{\rho}\right\|_{L^2} &\leqslant \left\|\partial^{\alpha_1}\overline{V}\right\|_{L^\infty}\|\nabla\partial^{\alpha_2}\widetilde{\rho}\|_{L^2}\\ &\leqslant C\left|\int(\rho_0+\omega_0)\right|\|\nabla g*\partial^{\alpha_1}\chi\|_{L^\infty}\|\widetilde{\rho}\|_{H^s}\\ &\leqslant C\left|\int(\rho_0+\omega_0)\right|\|\widetilde{\rho}\|_{H^s}\end{aligned}$$

by Claim (3) of Proposition 2.2.5. Thus we have the estimate we wanted to prove.

Now let us prove the last point of our theorem. Substracting the two continuity equations satisfied by $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$, we have

$$\partial_t(\widetilde{\omega}_1-\widetilde{\omega}_2)+\operatorname{div}(V_1(\widetilde{\omega}_1-\widetilde{\omega}_2)) = (V_2-V_1)\cdot\nabla\widetilde{\omega}_2.$$

Using estimate (2.2.10), we have

$$\begin{aligned}\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{L^\infty_T L^2} &\leqslant c\int_0^T\|(V_1-V_2)\cdot\nabla\widetilde{\omega}_2\|_{L^2}(\tau)\,\mathrm{d}\tau\\ &\leqslant CT\|V_1-V_2\|_{L^\infty_T L^2}\|\widetilde{\omega}_2\|_{H^3}.\end{aligned}$$

Now we prove the last estimate we need for $\widetilde{\rho}_1-\widetilde{\rho}_2$:

$$\partial_t(\widetilde{\rho}_1-\widetilde{\rho}_2)+\operatorname{div}(v_1(\widetilde{\rho}_1-\widetilde{\rho}_2)) = (v_2-v_1)\cdot\nabla\widetilde{\rho}_2+\operatorname{div}(v_2-v_1)\widetilde{\rho}_2.$$

We can bound the second term the same way that we did for the previous one:

$$\begin{aligned}\|(v_2-v_1)\cdot\nabla\widetilde{\rho}_2+\operatorname{div}(v_2-v_1)\widetilde{\rho}_2\|_{L^2} &\leqslant \|v_1-v_2\|_{L^2}\|\nabla\widetilde{\rho}_2\|_{L^\infty}\\ &\quad+\|\operatorname{div}(v_1-v_2)\|_{L^2}\|\widetilde{\rho}_2\|_{L^\infty}\\ &\leqslant 2\|v_1-v_2\|_{H^1}\|\widetilde{\rho}_2\|_{H^3}.\end{aligned}$$

Thus by (2.2.10),

$$\begin{aligned}\|\widetilde{\rho}_1-\widetilde{\rho}_2\|_{L^\infty_T L^2} &\leqslant CT\|\widetilde{\rho}_2\|_{L^\infty_T H^3}\|v_2-v_1\|_{L^\infty_T H^1}e^{cT\|\operatorname{div}(v_1)\|_{L^\infty_T H^2}}\\ &\leqslant CT\|\widetilde{\rho}_2\|_{L^\infty_T H^3}\|v_2-v_1\|_{L^\infty_T H^1}e^{cT\|u_1\|_{L^\infty_T H^3}}\end{aligned}$$

because $\operatorname{div}(v_1) = \operatorname{div}(u_1)$. □

Now let us prove Lemma 2.2.12:

Proof of Lemma 2.2.12. Solving the continuity equation by characteristics, we see that

$$\mathrm{supp}(\mu(t)) = \psi^t(\mathrm{supp}(\mu(0)))$$

where ψ is the flow associated to a. Moreover, for $x \in \mathrm{supp}\,\mu_0$,

$$\begin{aligned}
|\psi^t(x)| &\leqslant |\psi^0(x)| + |\psi^t(x) - \psi^0(x)| \\
&\leqslant |x| + \left| \int_0^t a(\tau, \psi^\tau(x))\,\mathrm{d}\tau \right| \\
&\leqslant R[\mu_0] + T\,\|a\|_{L^\infty_T L^\infty}.
\end{aligned}$$

Taking the supremum for all x in $\mathrm{supp}(\mu_0)$, we get (2.2.9).

□

2.2.4 Monokinetic spray System

In this section we prove the well-posedness result of system (2.1.2), that is Theorem 2.2.1:

Proof of Theorem 2.2.1. Let $(\rho_0, \omega_0) \in H^s$, $u_0 \in H^{s+1}$ and χ be a smooth function with compact support such that $\displaystyle\int_{\mathbb{R}^2} \chi = 1$. We recall that we have defined

$$M_0 := \max(\|\rho_0\|_{H^s}, \|\omega_0\|_{H^s}, \|u_0\|_{H^{s+1}}),\ R_0 := R[\rho_0 + \omega_0]$$

and

$$\begin{aligned}
X_T := \Big\{ &(\omega, \rho) \in L^\infty_T H^s \cap C_T H^{s-1} \Big| \omega(0) = \omega_0, \rho(0) = \rho_0, \\
&\|\rho\|_{L^\infty_T H^s} \leqslant 2M_0, \|\omega\|_{L^\infty_T H^s} \leqslant 2M_0, R_T[\rho + \omega] \leqslant 2R_0, \\
&\forall t \in [0,T], \int (\rho(t) + \omega(t)) = \int (\rho_0 + \omega_0), \\
&\forall t, t' \in [0,T], \|\rho(t) - \rho(t')\|_{H^{s-1}} \leqslant L|t - t'|, \\
&\|\omega(t) - \omega(t')\|_{H^{s-1}} \leqslant L|t - t'|, \Big\}
\end{aligned}$$

with $L > 0$ that we will fix later. Let us justify that X_T is a complete metric space for the distance

$$d((\rho_1, \omega_1), (\rho_2, \omega_2)) := \|\rho_1 - \rho_2\|_{L^\infty_T L^2} + \|\omega_1 - \omega_2\|_{L^\infty_T L^2}.$$

It is sufficient to prove that X_T is closed in $(L^\infty_T L^2)^2$. Let us consider a sequence of functions (ρ_N, ω_N) in X_T and $(\rho, \omega) \in (L^\infty_T L^2)^2$ such that

$$d((\rho_N, \omega_N), (\rho, \omega)) \underset{N\to\infty}{\longrightarrow} 0$$

and prove that $(\rho,\omega)\in X_T$. By Banach-Alaoglu's theorem, since H^s is a Hilbert space, for almost every time there exists a subsequence $\rho_{\varphi_t(N)}(t)$ that converges weakly in H^s. By uniqueness of the limit in weak-L^2, $\rho_N(t)$ converges weakly to $\rho(t)$ for almost every $t\in[0,T]$. By lower semi-continuity of the H^s norm we get that

$$\|\rho\|_{L^\infty_T H^s} \leqslant 2M_0. \tag{2.2.11}$$

By the same argument we can prove that

$$\|\omega\|_{L^\infty_T H^s} \leqslant 2M_0. \tag{2.2.12}$$

Now for almost every $t,t'\in[0,T]$, we know that $(\rho_N(t))$ and $(\rho_N(t'))$ converge weakly to $\rho(t)$ and $\rho(t')$ in H^s. By lower semi-continuity of the H^{s-1} norm we get that

$$\left\|\rho(t)-\rho(t')\right\|_{H^{s-1}} \leqslant L|t-t'|$$

and therefore ρ has a representative which is continuous in time with values in H^{s-1}. Using the same argument for ω we get that for all $t,t'\in[0,T]$

$$\begin{aligned}\left\|\rho(t)-\rho(t')\right\|_{H^{s-1}} &\leqslant L|t-t'|\\ \left\|\omega(t)-\omega(t')\right\|_{H^{s-1}} &\leqslant L|t-t'|.\end{aligned} \tag{2.2.13}$$

By time continuity of ρ and ω with value in H^{s-1} we get that

$$\begin{aligned}\omega(0)&=\omega_0\\ \rho(0)&=\rho_0.\end{aligned} \tag{2.2.14}$$

Moreover for all $t\in[0,T]$,

$$\int \mathbf{1}_{B(0,2R_0)}(\rho_N^2(t)+\omega_N^2(t)) \underset{N\to+\infty}{\longrightarrow} \int \mathbf{1}_{B(0,2R_0)}(\rho^2(t)+\omega^2(t)) = 0$$

by strong convergence in L^2. Thus ρ and ω have compact support and

$$R[\rho+\omega] \leqslant 2R_0. \tag{2.2.15}$$

Finally, uniform compact support and convergence in L^2 implies convergence in L^1 so we get that for every $t\in[0,T]$,

$$\int(\rho(t)+\omega(t)) = \int(\rho_0+\omega_0). \tag{2.2.16}$$

Inequalities (2.2.11), (2.2.12), (2.2.13), (2.2.14), (2.2.15) and (2.2.16) gives us that $(\rho,\omega)\in X_T$, so X_T is closed in $L^\infty_T L^2$.

Now let us build a contraction $X_T \longrightarrow X_T$. For $(\rho,\omega)\in X_T$ fixed, we have defined

— $V := -\nabla^\perp g*(\rho+\omega)$,

— $\overline{V} := -\left(\int \rho_0 + \omega_0\right) \nabla^\perp g * \chi$,

— $f := (V - \overline{V})^\perp - (\overline{V} \cdot \nabla)\overline{V}$.

By Corollary 2.2.8, $f \in L^\infty_T H^{s+1} \cap \mathcal{C}^0_T H^s$. Let T_1 be sufficiently small so that Theorem 2.2.10 can be applied and u be the solution of (2.2.6) given by this theorem, $v = u + \overline{V}$, and $(\widetilde{\rho}, \widetilde{\omega})$ be the solution of (2.2.8) given by Theorem 2.2.11. According to Theorem 2.2.10, the smallness of T_1 depends on M_0 and R_0. Now let us justify that for small enough $T_2 \leqslant T_1$, we have $(\widetilde{\rho}, \widetilde{\omega}) \in X_{T_2}$. By Theorem 2.2.10, we have the following estimates:

$$\|\widetilde{\rho}\|_{L^\infty_{T_1} H^s} \leqslant \|\rho_0\|_{H^s} e^{cT_1 \|u\|_{L^\infty_{T_1} H^s}} \exp\left(c e^{cT_1 \|u\|_{L^\infty_{T_1} H^s}} T_1 \|\nabla v\|_{L^\infty_{T_1} H^s}\right)$$
$$\|\widetilde{\omega}\|_{L^\infty_{T_1} H^s} \leqslant \|\omega_0\| e^{cT_1 \|\nabla V\|_{L^\infty_{T_1} H^s}}.$$

Remark that

$$\|\nabla V\|_{L^\infty_{T_1} H^s} \leqslant C \|\rho + \omega\|_{L^\infty_T H^s} \leqslant 4CM_0$$

by Claim (2) of Proposition 2.2.5. Moreover by Claim (2) of Proposition 2.2.5 and Theorem 2.2.10,

$$\begin{aligned} \|\nabla v\|_{L^\infty_{T_1} H^s} &\leqslant C(\|u\|_{L^\infty_{T_1} H^{s+1}} + \|\nabla \overline{V}\|_{L^\infty_{T_1} H^s}) \\ &\leqslant C(2M_0 + CR_0 M_0) \\ &\leqslant C(1 + R_0) M_0. \end{aligned}$$

Thus $\|\widetilde{\rho}\|_{L^\infty_{T_2} H^s} \leqslant 2M_0$ and $\|\widetilde{\omega}\|_{L^\infty_{T_2} H^s} \leqslant 2M_0$ if $T_2 \leqslant T_1$ and T_2 small enough with respect to M_0 and R_0. Now, by Lemma 2.2.12 and Claim (3) of Proposition 2.2.5, if $0 \leqslant t \leqslant T_2$ we have

$$\begin{aligned} R[\widetilde{\rho}(t) + \widetilde{\omega}(t)] &\leqslant R[\widetilde{\rho}(t)] + R[\widetilde{\omega}(t)] \\ &\leqslant R_0 + t(\|v\|_{L^\infty} + \|V\|_{L^\infty}) \\ &\leqslant R_0 + t(\|u\|_{H^s} + \|\overline{V}\|_{L^\infty} + \|V\|_{L^\infty}) \\ &\leqslant R_0 + t\left(2M_0 + C\left|\int \rho_0 + \omega_0\right| + CR_{T_2}[\rho + \omega]^{\frac{1}{3}} \|\rho + \omega\|_{H^1}\right) \\ &\leqslant R_0 + T_2(2M_0 + 2CR_0 M_0 + 4CR_0^{\frac{1}{3}} M_0) \\ &\leqslant 2R_0 \end{aligned}$$

if T_2 is small enough with respect to R_0 and M_0. By Theorem 2.2.11, we have:

$$\begin{aligned} \|\partial_t \widetilde{\omega}\|_{L^\infty_{T_2} H^{s-1}} &\leqslant C(1 + R_{T_2}[\rho + \omega]^{\frac{1}{3}}) \|\rho + \omega\|_{L^\infty_{T_2} H^s} \|\widetilde{\omega}\|_{L^\infty_{T_2} H^s} \\ &\leqslant C(1 + (2R_0)^{\frac{1}{3}}) 4M_0 2M_0 \end{aligned}$$

$$\|\partial_t\widetilde{\rho}\|_{L^\infty_{T_2}H^{s-1}} \leqslant C\left(\left|\int \rho_0+\omega_0\right| + \|u\|_{L^\infty_{T^2}H^{s+1}}\right)\|\widetilde{\rho}\|_{L^\infty_{T_2}H^s}$$
$$\leqslant C\big(2R_0M_0+2M_0\big)2M_0.$$

Choosing L large enough (with respect to M_0 and R_0), we have

$$\|\partial_t\widetilde{\omega}\|_{L^\infty_{T_2}H^{s-1}} \leqslant L$$
$$\|\partial_t\widetilde{\rho}\|_{L^\infty_{T_2}H^{s-1}} \leqslant L.$$

Thus we have built a map $\Phi : (\rho,\omega) \mapsto (\widetilde{\rho},\widetilde{\omega})$ such that $\Phi(X_{T_2}) \subset X_{T_2}$. We will now prove that Φ is a contraction for the $L^\infty_{T_2}L^2$ norm.

Let $(\rho_1,\omega_1),(\rho_2,\omega_2) \in X_{T_2}$, $(\widetilde{\rho}_1,\widetilde{\omega}_1) = \Phi(\rho_1,\omega_1)$ and $(\widetilde{\rho}_2,\widetilde{\omega}_2) = \Phi(\rho_2,\omega_2)$. By Theorem 2.2.11, we have

$$\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{L^\infty_{T_2}L^2} \leqslant CT_2\,\|V_1-V_2\|_{L^\infty_{T_2}L^2}\,\|\widetilde{\omega}_2\|_{L^\infty_{T_2}H^3}$$

and

$$\|\widetilde{\rho}_1-\widetilde{\rho}_2\|_{L^\infty_{T_2}L^2} \leqslant CT_2\,\|\widetilde{\rho}_2\|_{L^\infty_{T_2}H^3}\,\|v_2-v_1\|_{L^\infty_{T_2}H^1}\,e^{cT\|u_1\|_{L^\infty_{T_2}H^3}}.$$

Moreover, by Theorem 2.2.10:

$$\|v_2-v_1\|_{L^\infty_{T_2}H^1} = \|u_2-u_1\|_{L^\infty_{T_2}H^1} \leqslant Ce^{C(M_0,R_0)T_2}\,\|V_1-V_2\|_{L^\infty_{T_2}H^1}$$

Thus

$$\begin{aligned}
&\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{L^\infty_{T_2}L^2} + \|\widetilde{\rho}_1-\widetilde{\rho}_2\|_{L^\infty_{T_2}L^2}\\
&\qquad\leqslant 2CT_2M_0\,\|V_1-V_2\|_{L^\infty_{T_2}L^2}\\
&\qquad\quad + 2CT_2M_0e^{2cT_2M_0}Ce^{C(M_0,R_0)T_2}\,\|V_1-V_2\|_{L^\infty_{T_2}H^1}\\
&\qquad\leqslant C(M_0,R_0,T)T_2\,\|V_1-V_2\|_{L^\infty_{T_2}H^1}
\end{aligned}$$

for any $T_2 \leqslant T_1 \leqslant T$. Moreover,

$$\begin{aligned}
&\|V_1-V_2\|_{L^\infty_{T_2}H^1}\\
&\leqslant \|V_1-V_2\|_{L^\infty_{T_2}L^2} + \|\nabla(V_1-V_2)\|_{L^\infty_{T_2}L^2}\\
&\leqslant C(1+(1+(4R_0)^2)\,\|\rho_1+\omega_1-\rho_2-\omega_2\|_{L^\infty_{T_2}L^2}
\end{aligned}$$

by Claims (2) and (5) of Proposition 2.2.5. Thus Φ is a contraction if T_2 is small enough (with respect to M_0 and R_0), so it has a unique fixed point $(\rho,\omega) \in X_{T_2}$. $\square$

2.3 Mean-field limit

In this section we prove Proposition 2.1.5, Theorem 2.1.6, Proposition 2.1.9 and Proposition 2.1.12. Let us begin by proving Proposition 2.1.5.

2.3.1 Proof of Proposition 2.1.5

For $0 < \eta < 1$ we define

$$g^{(\eta)}(x) = \begin{cases} -\dfrac{1}{2\pi}\ln(\eta) & \text{if } |x| \leqslant \eta \\ g(x) & \text{if } |x| \geqslant \eta \end{cases}$$

and we denote $\delta_y^{(\eta)}$ the uniform probability measure on the circle of center y and radius η. We have the following lemma:

Lemma 2.3.1. *For any $0 < \eta < 1$ and $y \in \mathbb{R}^2$,*

$$\int g(x-z)\,\mathrm{d}\delta_y^{(\eta)}(z) = g^{(\eta)}(x-y)$$

Proof. By a change of variable we may assume that $y = 0$. The function

$$f(x) = \int_{\partial B(0,\eta)} g(x-z)\,\mathrm{d}\delta_0^{(\eta)}(z)$$

is locally bounded and satisfies $\Delta f = -\delta_0^{(\eta)} = \Delta g^{(\eta)}$. Now if $|x| \geqslant \eta$, we have

$$\begin{aligned}\int_{\partial B(0,\eta)} g(x-z)\,\mathrm{d}\delta_0^{(\eta)}(z) - g^{(\eta)}(x) &= \int_{\partial B(0,\eta)} (g(x-z)-g(x))\,\mathrm{d}\delta_0^{(\eta)}(z) \\ &= \int_{\partial B(0,\eta)} g\left(\frac{x}{|x|} - \frac{z}{|x|}\right)\mathrm{d}\delta_0^{(\eta)}(z) \\ &\underset{|x|\to\infty}{\longrightarrow} \int_{\partial B(0,\eta)} -\frac{1}{2\pi}\ln(1) = 0\end{aligned}$$

by dominated convergence theorem. Therefore $f - g^{(\eta)}$ is a harmonic bounded function so it is constant. Since $f(z) = g(\eta) = g^{(\eta)}(z)$ for any z of norm η, we get that $f = g^{(\eta)}$. ☐

Integrating by parts, since $\displaystyle\int_{\mathbb{R}^2} \omega - \omega_N = 0$, we have

(2.3.1)
$$\|\nabla g * (\omega - \omega_N)\|_{L^2}^2 = \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y.$$

For a more detailed justification of such integrations by parts we refer to [81, Equality (1.23)]. Therefore we only need to justify Inequality (2.1.4) to get that $\mathcal{H} \geqslant 0$. For that purpose we define

$$\rho_N^{(\eta)} := \frac{1}{N}\sum_{i=1}^N \delta_{q_i}^{(\eta)}.$$

We have

$$\begin{aligned}
\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} & g(x-y)(\rho+\omega-\rho_N-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\
=& \iint_{(\mathbb{R}^2\times\mathbb{R}^2)} g(x-y)(\rho+\omega-\rho_N^{(\eta)}-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\
&+ \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\,\mathrm{d}\rho_N(x)\,\mathrm{d}\rho_N(y) - \,\mathrm{d}\rho_N^{(\eta)}(x)\,\mathrm{d}\rho_N^{(\eta)}(y)) \\
&+ 2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)} g(x-y)(\omega-\omega_N+\rho)(x)\,\mathrm{d}x\,\mathrm{d}(\rho_N^{(\eta)}-\rho_N)(y) \\
=& L_1+L_2+L_3.
\end{aligned}$$

Integrating by parts the first line we find that

$$L_1 = \int |\nabla g*(\rho+\omega-\rho_N^{(\eta)}-\omega_N)|^2 \geqslant 0.$$

For the second line, by Lemma 2.3.1 we have

$$L_2 = \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\int_{\mathbb{R}^2}(g(q_i-q_j)-g^{(\eta)}(q_i-y))\,\mathrm{d}(\delta_{q_i}+\delta_{q_i}^{(\eta)})(y)\,\mathrm{d}y$$

This quantity have been bounded in [71, Inequality 2.14] so we get

$$L_2 \geqslant -\frac{C}{N}\sum_{i=1}^N \eta^2.$$

Finally,

$$|L_3| \leqslant C|g*(\omega-\omega_N+\rho)|_{C^{0,\gamma}}\eta^\gamma$$

so by Morrey's inequality (see for example [14, Theorem 9.12]) and Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]) for some $p>2$ we have

$$\begin{aligned}
|L_3| &\leqslant C\left\|\nabla g*(\omega-\omega_N+\rho)\right\|_{L^p}\eta^\gamma \\
&\leqslant C\left\|\omega-\omega_N+\rho\right\|_{L^{\frac{2p}{p+2}}} \\
&\leqslant C(\|\omega\|_{L^1\cap L^\infty}+\|\omega_N\|_{L^1\cap L^\infty}+\|\rho\|_{L^1\cap L^\infty})\eta^\gamma.
\end{aligned}$$

We get Inequality (2.1.4) by taking $\eta = N^{-1}$.

2.3.2 Proof of Theorem 2.1.6

We want to compute the derivative of the functional $\mathcal{H}_N = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$ defined in (2.1.6). We will denote $\alpha := \omega + \rho$ and $\alpha_N := \omega_N + \rho_N$.

$$\begin{aligned}
T_1 &:= \frac{1}{N}\sum_{i=1}^N |v(q_i) - p_i|^2 \\
T_2 &:= \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\alpha_N(t,\,\mathrm{d}x)\alpha_N(t,\,\mathrm{d}y) \\
T_3 &:= \int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\alpha(t,x)\alpha(t,y)\,\mathrm{d}x\,\mathrm{d}y \\
T_4 &:= -2\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\alpha(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\
T_5 &:= \|\omega(t) - \omega_N(t)\|_{L^2}^2 \\
T_6 &:= \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y \\
T_7 &= BN^{-\gamma}.
\end{aligned}$$

Claim 2.3.2. For every $t \in [0,T]$, we have

$$\begin{aligned}
\frac{\mathrm{d}T_1}{\mathrm{d}t} =& -\frac{2}{N}\sum_{i=1}^N \nabla v(q_i) : (v(q_i)-p_i)^{\otimes 2} \\
& -\frac{2}{N}\sum_{i=1}^N p_i \cdot \left(\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N \nabla g(q_i-q_j) + \nabla g * (\omega_N - \alpha)(q_i)\right) \\
& + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\rho_N(t,\,\mathrm{d}x)(\alpha_N-\alpha)(t,\,\mathrm{d}y) \\
=:&\ T_{1,1} + T_{1,2} + T_{1,3}.
\end{aligned}$$

Proof of Claim 2.3.2. Since $v \in \mathcal{C}^1([0,T]\times\mathbb{R}^2,\mathbb{R}^2)$ we can compute

$$\begin{aligned}
\frac{\mathrm{d}T_1}{\mathrm{d}t} =& \frac{2}{N}\sum_{i=1}^N (v(q_i)-p_i)\cdot(\partial_t v(q_i) + (p_i\cdot\nabla)v(q_i) - \dot{p}_i) \\
=& \frac{2}{N}\sum_{i=1}^N (v(q_i)-p_i)\cdot\Big(-(v\cdot\nabla)v(q_i) + (v-V)^\perp(q_i) \\
& + (p_i\cdot\nabla)v(q_i) - p_i^\perp + \nabla g * \omega_N(q_i) + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N \nabla g(q_i-q_j)\Big)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{2}{N}\sum_{i=1}^{N}(v(q_i)-p_i)\cdot\Big(((p_i-v(t,q_i))\cdot\nabla)v(q_i)+(v(q_i)-p_i)^{\perp}\\
&+\nabla g*(\omega_N-\alpha)(q_i)+\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\nabla g(q_i-q_j)\Big)\\
=&-\frac{2}{N}\sum_{i=1}^{N}\nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&+\frac{2}{N}\sum_{i=1}^{N}(v(q_i)-p_i)\cdot\Big(\nabla g*(\omega_N-\alpha)(q_i)+\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\nabla g(q_i-q_j)\Big)\\
=&-\frac{2}{N}\sum_{i=1}^{N}\nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&-\frac{2}{N}\sum_{i=1}^{N}p_i\cdot\Big(\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\Big)\\
&+\frac{2}{N}\sum_{i=1}^{N}v(q_i)\cdot\Big(\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\Big)\\
=&-\frac{2}{N}\sum_{i=1}^{N}\nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&-\frac{2}{N}\sum_{i=1}^{N}p_i\cdot\Big(\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\nabla g(q_i-q_j)+\nabla g*(\omega_N-\alpha)(q_i)\Big)\\
&+2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}v(t,x)\cdot\nabla g(x-y)\rho_N(t,\,\mathrm{d}x)\,\mathrm{d}(\alpha_N-\alpha)(t,y)\\
=&T_{1,1}+T_{1,2}+T_{1,3}.
\end{aligned}
$$

□

In the incoming computations, we will find some terms which look like $T_{1,2}$, that is, terms depending on p_i (which will cancel out) or like $T_{1,3}$, that is terms of the form:

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}A(t,x)\cdot\nabla g(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

with A a smooth vector field (for example v or V) and μ, ν some signed finite measures (for example α or $\rho-\rho_N$). We will finish our computations grouping all terms corresponding to the same vector field A. Let us now compute the time derivative of T_2. Notice that the energy

$$\mathcal{E}_N = \frac{1}{N}\sum_{i=1}^{N}|p_i|^2 + \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)\,\mathrm{d}\alpha_N(t,x)\,\mathrm{d}\alpha_N(t,y)$$

of System (2.1.1) is constant in time (for more details see [52, Proposition 5.1]). Thus we have

$$T_2 = \mathcal{E}_N - \frac{1}{N}\sum_{i=1}^{N}|p_i|^2$$

and

$$\begin{aligned}\frac{\mathrm{d}T_2}{\mathrm{d}t} &= -\frac{2}{N}\sum_{i=1}^{N} p_i\cdot\Big(p_i^{\perp} - \nabla g * \omega_N(q_i) - \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(q_i-q_j)\Big)\\ &= \frac{2}{N}\sum_{i=1}^{N} p_i\cdot\Big(\nabla g * \omega_N(q_i) + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\nabla g(q_i-q_j)\Big) \qquad (2.3.2)\\ &=: T_{2,1}.\end{aligned}$$

Let us compute the time derivative of the third term:

Claim 2.3.3. $T_3 \in W^{1,\infty}([0,T])$ and for almost every $t\in[0,T]$, we have

$$\begin{aligned}\frac{\mathrm{d}T_3}{\mathrm{d}t} &= 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} v(t,x)\cdot\nabla g(x-y)\rho(t,x)\alpha(t,y)\,\mathrm{d}x\,\mathrm{d}y.\\ &=: T_{3,1}.\end{aligned}$$

Proof of Claim 2.3.3. Let $(g_\eta)_{0<\eta<1}$ be a family of smooth functions such that

— $g_\eta(x) = g(x)$ if $|x|\geqslant\eta$,
— $|g_\eta(x)|\leqslant|g(x)|$,
— $|\nabla g_\eta(x)|\leqslant\dfrac{C}{|x|}$.
— $g_\eta(-x) = g_\eta(x)$.

For $0\leqslant s,t\leqslant T$ and $0<\eta<1$, we have

$$T_3(t)-T_3(s) = \iint g(x-y)(\alpha(t,x)\alpha(t,y)-\alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y.$$

Remark that

$$\begin{aligned}&\iint |g_\eta(x-y)(\alpha(t,x)\alpha(t,y)-\alpha(s,x)\alpha(s,y))|\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad\leqslant \iint |g(x-y)|(\alpha(t,x)\alpha(t,y)-\alpha(s,x)\alpha(s,y))|\,\mathrm{d}x\,\mathrm{d}y < +\infty\end{aligned}$$

because α has compact support. Thus by dominated convergence theorem we have

(2.3.3)
$$T_3(t) - T_3(s) = \lim_{\eta\to 0} \iint g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y.$$

Since g_η is smooth and α has compact support, we have by (2.1.3) that

$$\int g_\eta(x-y)(\alpha(t,x)-\alpha(s,x))\,\mathrm{d}x = \int_s^t \int (\rho v + \omega V)(\tau,x)\cdot\nabla g_\eta(x-y)\,\mathrm{d}x\,\mathrm{d}\tau$$

Since $(\rho v+\omega V)*\nabla g_\eta \in L^\infty([0,T],\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}))$, we get from the upper equation that $g_\eta * \alpha \in W^{1,\infty}([0,T],\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}))$ and that for almost every $t\in[0,T]$,

$$\partial_t(g_\eta * \alpha) = -(\rho v + \omega V) * \nabla g_\eta.$$

Thus we can use $g_\eta * \alpha$ as a test function in (2.1.3) to get

$$\begin{aligned}
&\iint g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = -\int_s^t \int ((\rho v + \omega V) * \nabla g_\eta)\alpha + \int_s^t \int (\rho v + \omega V)\cdot\nabla(g_\eta * \alpha) \\
&\quad = -\int_s^t \iint (\rho v + \omega V)(\tau,x)\cdot\nabla g_\eta(y-x)\alpha(\tau,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&\quad + \int_s^t \iint (\rho v + \omega V)(\tau,x)\cdot\nabla g_\eta(x-y)\alpha(\tau,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau \\
&\quad = 2\int_s^t \iint (\rho v + \omega V)(\tau,x)\cdot\nabla g_\eta(x-y)\alpha(\tau,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}$$

Remark that for almost any $\tau\in[0,T]$, we have

$$\begin{aligned}
&\iint |\nabla g_\eta(x-y)\cdot(\rho v + \omega V)(\tau,x)\alpha(\tau,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \iint \frac{1}{|x-y|}|(\rho v + \omega V)(\tau,x)|\alpha(\tau,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant \|\rho v + \omega V\|_{L^\infty_T L^1} \sup_{\tau\in[0,T]}\sup_{x\in\mathbb{R}^2} \int \frac{|\alpha(\tau,y)|}{|x-y|}\,\mathrm{d}y \\
&\quad \leqslant (\|\rho\|_{L^\infty_T L^1}\|v\|_{L^\infty_T L^\infty} + \|\omega\|_{L^\infty_T L^1}\|V\|_{L^\infty_T L^\infty}) R_T[\alpha]\,\|\alpha\|_{L^\infty_T L^\infty}.
\end{aligned}$$

where the last inequality follows from the proof of Claim (3) of Proposition 2.2.5. Thus by dominated convergence theorem,

$$\begin{aligned}
&\iint g_\eta(x-y)(\alpha(t,x)\alpha(t,y) - \alpha(s,x)\alpha(s,y))\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \underset{\eta\to 0}{\longrightarrow} 2\int_s^t \iint (\rho v + \omega V)(\tau,x)\cdot\nabla g(x-y)\alpha(\tau,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}$$

Combining the upper limit with (2.3.3) we get that

$$T_3(t) - T_3(s) = 2\int_s^t \iint (\rho v + \omega V)(\tau, x) \cdot \nabla g(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau.$$

Remark that since $\nabla g * \alpha = -V^{\perp}$, we have

$$\int_s^t \iint (\omega V)(\tau, x) \cdot \nabla g(x-y)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau = 0.$$

Finally, we get

$$T_3(t) - T_3(s) = 2\int_s^t \iint v(\tau, x) \cdot \nabla g(x-y)\rho(\tau, x)\alpha(\tau, y)\,\mathrm{d}x\,\mathrm{d}y$$

which ends the proof of Claim 2.3.3 for almost every $t \in [0, T]$. □

Now for the fourth term, we have:

Claim 2.3.4.

$$\begin{aligned}\frac{\mathrm{d}T_4}{\mathrm{d}t} =& 2\iint V(t,x) \cdot \nabla g(x-y)(\omega_N - \omega)(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ &- 2\iint v(t,x) \cdot \nabla g(x-y)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ &- \frac{2}{N}\sum_{i=1}^N p_i \cdot \nabla g * \alpha(q_i) \\ =&: T_{4,1} + T_{4,2} + T_{4,3}.\end{aligned}$$

Proof of Claim 2.3.4. Recall that

$$\begin{aligned}T_4 =& -2\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\alpha(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ =& -2\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}\,\mathrm{d}y \\ &- \frac{2}{N}\sum_{i=1}^N \int_{\mathbb{R}} g(x - q_i(t))\alpha(t,x)\,\mathrm{d}x.\end{aligned}$$

Using g_η in the same way we did for the previous claim, one can prove that T_4 is $W^{1,\infty}$ and that for almost every $t \in (0, T)$,

$$\begin{aligned}\frac{\mathrm{d}T_4}{\mathrm{d}t} =& \Big(-2\iint \nabla g(x-y) \cdot (V(t,x)\omega(t,x) + \rho(t,x)v(t,x))\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y) \\ &+ 2\iint \nabla g(x-y) \cdot V_N(t,y)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{N}\sum_{i=1}^{N} p_i\cdot\int \nabla g(x-q_i)\alpha(t,x)\,\mathrm{d}x\\
=&A_1+A_2.
\end{aligned}
$$

Let us compute each term. For the second term in A_1, remark that:

$$
\begin{aligned}
&\iint \nabla g(x-y)\cdot V_N(t,y)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\\
&=-\iiint \nabla g(x-y)\cdot\nabla^{\perp}g(y-z)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z)\\
&=\iiint \nabla^{\perp}g(x-y)\cdot\nabla g(y-z)\alpha(t,x)\omega_N(t,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z)\\
&=\iint\left(-\int\nabla^{\perp}g(y-x)\alpha(t,x)\,\mathrm{d}x\right)\cdot\nabla g(y-z)\omega_N(t,y)\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z)\\
&=\iint V(t,y)\cdot\nabla g(y-z)\omega_N(t,y)\,\mathrm{d}y\,\mathrm{d}\alpha_N(t,z)\\
&=\iint V(t,x)\cdot\nabla g(x-y)\omega_N(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
A_1=&2\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y)\\
&-2\iint v(t,x)\cdot\nabla g(x-y)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y)\\
=&T_{4,1}+T_{4,2}.
\end{aligned}
$$

For the second term:

$$
\begin{aligned}
A_2&=-\frac{2}{N}\sum_{i=1}^{N}p_i\cdot\int\nabla g(q_i-x)\alpha(t,x)\,\mathrm{d}x\\
&=-\frac{2}{N}\sum_{i=1}^{N}p_i\cdot\nabla g*\alpha(q_i)\\
&=T_{4,3}.
\end{aligned}
$$

□

We now need to differentiate the fifth term with respect to time.

Claim 2.3.5. T_5 is Lipschitz and for almost every $t\in[0,T]$ we have:

$$
\begin{aligned}
\frac{\mathrm{d}T_5}{\mathrm{d}t}&=-2\int\nabla\omega\cdot(V-V_N)(\omega-\omega_N)\\
&=:T_{5,1}.
\end{aligned}
$$

Proof of Claim 2.3.5. Let $(\chi_\eta)_{\eta>0}$ be a sequence of mollifiers with compact support and set $\omega_N^\eta = \chi_\eta * \omega_N$. For $t, s \in [0,T]$, we have

$$\begin{aligned} T_5(t) - T_5(s) &= \int |\omega(t) - \omega_N(t)|^2 - \int |\omega(s) - \omega_N(s)|^2 \\ &= \lim_{\eta\to 0} \int |\omega(t) - \omega_N^\eta(t)|^2 - \int |\omega(s) - \omega_N^\eta(s)|^2 \\ &= \lim_{\eta\to 0} \int_s^\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int |\omega(\tau) - \omega_N^\eta(\tau)|^2 \right) \mathrm{d}\tau \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \int |\omega(\tau) - \omega_N^\eta(\tau)|^2 =& 2\int (\omega - \omega_N^\eta) \operatorname{div}(\omega V - \chi_\eta * (\omega_N V_N)) \\ =& 2\int (\omega - \omega_N^\eta) \operatorname{div}(\omega(V - V_N) + (\omega - \omega_N^\eta) V_N) \\ &+ 2\int (\omega - \omega_N^\eta) \operatorname{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) \\ =& 2\int (\omega - \omega_N^\eta) \nabla\omega \cdot (V - V_N) \\ &+ 2\int (\omega - \omega_N^\eta) \nabla(\omega - \omega_N^\eta) \cdot V_N \\ &+ 2\int \omega \operatorname{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) \\ &- 2\int \omega_N^\eta \operatorname{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)). \end{aligned}$$

For the first term, remark that for any $1 < p < 2$, we have $V_N \in L^p_{\mathrm{loc}}$ and $\omega_N^\eta \underset{\eta\to 0}{\longrightarrow} \omega_N$ in $L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus since $\omega - \omega_N$ has compact support and $\nabla\omega \in L^\infty$, we have

$$2\int (\omega - \omega_N^\eta) \nabla\omega \cdot (V - V_N) \underset{\eta\to 0}{\longrightarrow} 2\int (\omega - \omega_N) \nabla\omega \cdot (V - V_N).$$

Remark also that the second term cancels out because V_N is divergent-free:

$$\int (\omega - \omega_N^\eta) V_N \cdot \nabla(\omega - \omega_N^\eta) = -\frac{1}{2} \int V_N \cdot \nabla |\omega - \omega_N^\eta|^2 = 0$$

We will now prove that the two last term tends to zero. For the third term, we integrate by parts to get:

$$2\int \omega \operatorname{div}(\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) = -2\int \nabla\omega \cdot (\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N))$$

Since $\omega_N^\eta V_N \underset{\eta\to 0}{\longrightarrow} \omega_N V_N$, $\chi_\eta * (\omega_N V_N) \underset{\eta\to 0}{\longrightarrow} \omega_N V_N$ in L^1 and $\nabla\omega \in L^\infty$, we have

$$2\int \nabla\omega \cdot (\omega_N^\eta V_N - \chi_\eta * (\omega_N V_N)) \underset{\eta\to 0}{\longrightarrow} 0.$$

For the last term, since all the q_i are outside of the support of ω_N (see Remark 2.1.3), they are also outside of the support of ω_N^η if η is small enough. Thus we have:

$$V_N \in \ W^{1,p}(\operatorname{supp} \omega_N^\eta)$$

for any $2 < p < +\infty$. By the commutator estimate of DiPerna and Lions in [24] (see [21, Lemma 2.2] for more details) we get

$$[V_N, \chi_\eta *]\omega_N \underset{\eta \to 0}{\longrightarrow} 0 \qquad \text{in } L^1_{\text{loc}}.$$

Since ω_N^η is uniformly bounded in L^∞, we obtain

$$\int \omega_N^\eta [V_N, \chi_\eta *]\omega_N \underset{\eta \to 0}{\longrightarrow} 0.$$

which ends the proof of Claim 2.3.5. □

For the sixth term:

Claim 2.3.6. T_6 Lipschitz and for almost every $t \in [0, T]$ we have

$$\begin{aligned}\frac{\mathrm{d}T_6}{\mathrm{d}t} &= 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot V(t,x)(\omega - \omega_N)(t,x)(\omega-\omega_N)(t,y) \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant 2 \int_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot \nabla^\perp g * (\omega - \omega_N)(t,x)\omega_N(t,x) \,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y).\end{aligned}$$

Proof of Claim 2.3.6. Using g_η in the same way we did for Claim 2.3.3, one can prove that T_6 is $W^{1,\infty}$ and that for almost every $t \in (0, T)$,

$$\begin{aligned}\frac{\mathrm{d}T_6}{\mathrm{d}t} =& 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot (V\omega - V_N\omega_N)(t,x)(\omega-\omega_N)(t,y) \,\mathrm{d}x\,\mathrm{d}y \\ =& 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot V(t,x)(\omega - \omega_N)(t,x)(\omega-\omega_N)(t,y) \,\mathrm{d}x\,\mathrm{d}y \\ &+ 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot (V - V_N)(t,x)\omega_N(t,x)(\omega-\omega_N)(t,y) \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Since $V - V_N = -\nabla^\perp g * (\omega - \omega_N + \rho - \rho_N)$ we get

$$\begin{aligned}&\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot (V - V_N)(t,x)\omega_N(t,x)(\omega-\omega_N)(t,y) \,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla g(x-y) \cdot \nabla^\perp g * (\omega - \omega_N)(t,x)\omega_N(t,x) \,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y)\end{aligned}$$

which ends the proof of Claim 2.3.6. □

Remark that all terms depending on p_i (coming from the equations of Claim 2.3.2, Claim 2.3.4 and Equation (2.3.2)) cancels out, that is

$$T_{1,2} + T_{2,1} + T_{4,3} = 0.$$

Now let us group all terms of the form

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

coming from the equations of Claims 2.3.2, 2.3.3 and 2.3.4:

$$\begin{aligned}
&T_{1,3} + T_{3,1} + +T_{4,2}\\
=&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)(\rho_N\otimes(\alpha_N-\alpha)-\rho\otimes\alpha_N+\rho\otimes\alpha)(\,\mathrm{d}x\,\mathrm{d}y)\\
=&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\rho-\rho_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
=&2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&-2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)(\omega-\omega_N)(t,x)\,\mathrm{d}x\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
=&\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}(v(t,x)-v(t,y))\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&+2\int v^\perp(t,x)\cdot(V-V_N)(t,x)(\omega-\omega_N)(t,x)\,\mathrm{d}x
\end{aligned}$$

because

$$\begin{aligned}
2&\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,x)\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} v(t,y)\cdot\nabla g(y-x)\,\mathrm{d}(\alpha-\alpha_N)(t,y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}(v(t,x)-v(t,y))\cdot\nabla g(x-y)\,\mathrm{d}(\alpha-\alpha_N)(t,x)\,\mathrm{d}(\alpha-\alpha_N)(t,y).
\end{aligned}$$

Let us do the same for V (there is only one term, coming from the equations of Claim 2.3.4):

$$\begin{aligned}
T_{4,1} &= 2\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\,\mathrm{d}x\,\mathrm{d}\alpha_N(t,y)\\
&= 2\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\,\mathrm{d}x\,\mathrm{d}(\alpha-\alpha_N)(t,y)
\end{aligned}$$

because

$$\iint V(t,x)\cdot\nabla g(x-y)(\omega_N-\omega)(t,x)\alpha(t,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$=\int V(t,x)(\omega_N-\omega)(t,x)\cdot V^\perp(t,x)\,\mathrm{d}x$$
$$=0.$$

Thus

$$T_{4,1}=-2\iint V^\perp(t,x)\cdot(V-V_N)(t,x)(\omega(t,x)-\omega_N(t,x))\,\mathrm{d}x.$$

Putting all terms together, we obtain

(2.3.4)
$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t}\\
&=-\frac{2}{N}\sum_{i=1}^N\nabla v(q_i):(v(q_i)-p_i)^{\otimes 2}\\
&\quad+\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta}(v(t,x)-v(t,y))\cdot\nabla g(x-y)(\alpha-\alpha_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\
&\quad+\int_{\mathbb{R}^2}A\cdot(V-V_N)(\omega-\omega_N)\\
&\quad+2\iint_{\mathbb{R}^2\times\mathbb{R}^2}\nabla g(x-y)\cdot V(t,x)(\omega-\omega_N)(t,x)(\omega-\omega_N)(t,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+2\iint_{\mathbb{R}^2\times\mathbb{R}^2}\nabla g(x-y)\cdot\nabla^\perp g*(\omega-\omega_N)(t,x)\omega_N(t,x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y)\\
&=:R_1+R_2+R_3+R_4+R_5
\end{aligned}$$

with $A=2(v^\perp-V^\perp-\nabla\omega)$. In order to control R_3, we will need the following result:

Lemma 2.3.7. *If $A\in W^{1,\infty}$, then there exists $\lambda>0$ and a constant C depending only on $\|A\|_{W^{1,\infty}}$ such that*

$$\left|\int A\cdot(V-V_N)(\omega-\omega_N)\right|\leqslant C\Big(\mathcal{F}(Q_N,\rho)+\|\omega-\omega_N\|_{L^2}^2$$
$$+\iint g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y+N^{-\lambda}\Big)$$

where $\mathcal{F}$ is the functionnal defined by (2.1.14).

Proof. Let us fix $I=\int A\cdot(V-V_N)(\omega-\omega_N)$, then

$$I=-\iint A(x)\cdot\nabla^\perp g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}(\omega+\rho-\omega_N-\rho_N)(y)\,\mathrm{d}x$$

$$
\begin{aligned}
=&\frac{1}{2}\iint (A^\perp(x)-A^\perp(y))\cdot\nabla g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\\
&-\int \nabla g * [A^\perp(\omega-\omega_N)](y)\,\mathrm{d}(\rho-\rho_N)(y)\\
=:&I_1+I_2.
\end{aligned}
$$

By [81, Lemma 4.3],

$$
I_1 = c\int \nabla A^\perp : [g*(\omega-\omega_N), g*(\omega-\omega_N)]
$$

where for $i,j\in\{1,2\}$ and h regular enough,

$$
[h,h]_{i,j} = 2\partial_i h\partial_j h - |\nabla h|^2\delta_{i,j}.
$$

Hence

$$
|I_1| \leqslant C\,\|\nabla A\|_{L^\infty}\,\|\nabla g*(\omega-\omega_N)\|_{L^2}^2\,.
$$

Therefore by (2.3.1) we have

$$
|I_1| \leqslant C\,\|\nabla A\|_{L^\infty}\iint g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.3.5}
$$

Now denote

$$
\xi_N := -\nabla g * [A^\perp(\omega-\omega_N)].
$$

We can write

$$
I_2 = \int \xi_N(y)(\rho-\rho_N)(\,\mathrm{d}y).
$$

Using Proposition 2.1.13 (proved in [81]), we get that for any $0<\theta<1$, there exists constants $C,\lambda>0$ such that

$$
\begin{aligned}
|I_2| \leqslant& C|\xi_N|_{\mathcal{C}^{0,\theta}}N^{-\lambda}\\
&+C\,\|\nabla\xi_N\|_{L^2}\left(\mathcal{F}(Q_N,\rho)+(1+\|\rho\|_{L^\infty})N^{-1}+\frac{\ln(N)}{N}\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Morrey's inequality (see for example [14, Theorem 9.12]) and Proposition 2.2.7, for some $p>2$ depending only on θ, we have

$$
\begin{aligned}
|\xi_N|_{\mathcal{C}^{0,\theta}} &\leqslant C\left\|\nabla^2 g*[A^\perp(\omega-\omega_N)]\right\|_{L^p}\\
&\leqslant C\,\|A(\omega-\omega_N)\|_{L^p}\\
&\leqslant C\,\|A\|_{L^\infty}\,\|\omega-\omega_N\|_{L^p}\\
&\leqslant C\,\|A\|_{L^\infty}\left(\|\omega_0\|_{L^p}+\|\omega_{N,0}\|_{L^p}\right)
\end{aligned}
$$

by Remark 2.1.3. Therefore by Assumption (2.1.7),

$$
|\xi_N|_{\mathcal{C}^{0,\theta}} \leqslant C.
$$

where C is independent of N. Now, by Proposition 2.2.7,

$$\left\|\nabla^2 g * [A^\perp(\omega-\omega_N)]\right\|_{L^2} \leqslant \|A\|_{L^\infty} \|\omega-\omega_N\|_{L^2}.$$

Thus we obtain the inequality we wanted to prove. □

Let us get back to the expression of $\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t} = R_1 + R_2 + R_3 + R_4 + R_5$ given by (2.3.4). We have

$$|R_1| \leqslant \frac{2\|\nabla v\|_{L^\infty}}{N} \sum_{i=1}^N |v(q_i)-p_i|^2 \leqslant 2\|\nabla v\|_{L^\infty} \mathcal{H}_N. \tag{2.3.6}$$

For the second term,

$$\begin{aligned} R_2 = &\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,x)-v(t,y))\cdot\nabla g(x-y)(\omega-\omega_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,x)-v(t,y))\cdot\nabla g(x-y)(\rho-\rho_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y) \\ &+ 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (v(t,y)-v(t,x)) \\ &\cdot \nabla g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y) \\ =:&R_{2,1}+R_{2,2}+R_{2,3}. \end{aligned}$$

We can bound $R_{2,1}$ as we did to obtain Inequality (2.3.5) and we get

$$|R_{2,1}| \leqslant C\|\nabla v\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.3.7}$$

Using Proposition 2.1.15 (proved in [81]) with $\mu = \rho \in L^\infty$, we get

$$|R_{2,2}| \leqslant C\|v\|_{W^{1,\infty}} \left(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-\lambda}\right). \tag{2.3.8}$$

Now,

$$R_{2,3} = \int \chi_N \,\mathrm{d}(\rho-\rho_N)$$

with $\chi_N = -2v\cdot\nabla g*(\omega-\omega_N) + 2\nabla g*(v(\omega-\omega_N))$. Using Proposition 2.1.13, we get that

$$\begin{aligned} |R_{2,3}| \leqslant &C\Bigg(|\chi_N|_{\mathcal{C}^{0,\theta}} N^{-\lambda} + \|\nabla\chi_N\|_{L^2}\Bigg(\mathcal{F}(Q_N,\rho) \\ &+ (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\Bigg)^{\frac{1}{2}}\Bigg). \end{aligned}$$

Now by Morrey's inequality (see for example [14, Theorem 9.12]), Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]) and Proposition 2.2.7, for some $p > 2$ we have

$$\begin{aligned}|\chi_N|_{\mathcal{C}^{0,\theta}} \leqslant & C(\|\nabla v\|_{L^\infty} \|\nabla g * (\omega - \omega_N)\|_{L^p} + \|v\|_{L^\infty} \left\|\nabla^2 g * (\omega - \omega_N)\right\|_{L^p} \\ & + \left\|\nabla^2 g * (v(\omega - \omega_N))\right\|_{L^p}) \\ & \leqslant C \|v\|_{W^{1,\infty}} (\|\omega - \omega_N\|_{L^p} + \|\omega - \omega_N\|_{L^{\frac{2p}{p+2}}}) \\ & \leqslant C \|v\|_{W^{1,\infty}} (\|\omega - \omega_N\|_{L^1} + \|\omega - \omega_N\|_{L^\infty}) \\ & \leqslant C \|v\|_{W^{1,\infty}} (\|\omega_0\|_{L^1} + \|\omega_{N,0}\|_{L^1} + \|\omega_0\|_{L^\infty} + \|\omega_{N,0}\|_{L^\infty})\end{aligned}$$

by Remark 2.1.3. Therefore by Assumption (2.1.7),

$$|\chi_N|_{\mathcal{C}^{0,\theta}} \leqslant C.$$

Moreover, using Proposition 2.2.7 and Equation (2.3.1), we have

$$\begin{aligned}\|\nabla\chi_N\|_{L^2} \leqslant & \|\nabla v\|_{L^\infty} \|\nabla g * (\omega - \omega_N)\|_{L^2} + \|v\|_{L^\infty} \left\|\nabla^2 g * (\omega - \omega_N)\right\|_{L^2} \\ & + \left\|\nabla^2 g * (v(\omega - \omega_N))\right\|_{L^2} \\ & \leqslant C \|v\|_{W^{1,\infty}} \Bigg[\bigg(\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega - \omega_N)(x) \\ & \times (\omega - \omega_N)(y) \,\mathrm{d}x \,\mathrm{d}y \bigg)^{\frac{1}{2}} \\ & + \|\omega - \omega_N\|_{L^2} \Bigg].\end{aligned}$$

Therefore

$$\begin{aligned}|R_{2,3}| \leqslant & C \|v\|_{W^{1,\infty}} \bigg(\|\omega - \omega_N\|_{L^2}^2 \\ & + \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega - \omega_N)(x)(\omega - \omega_N)(y) \,\mathrm{d}x \,\mathrm{d}y \\ & + \mathcal{F}(Q_N, \rho) + (1 + \|\rho\|_{L^\infty}) N^{-\lambda} \bigg).\end{aligned} \tag{2.3.9}$$

Combining inequalities (2.3.7), (2.3.8) and (2.3.9) we find that

$$|R_2| \leqslant C \|v\|_{W^{1,\infty}} (\mathcal{H}_N + \mathcal{F}(Q_N, \rho) + (1 + \|\rho\|_{L^\infty}) N^{-\lambda}). \tag{2.3.10}$$

Now using Lemma 2.3.7, since V, v and $\nabla\omega$ are in L^∞,

$$\begin{aligned}|R_3| \leqslant & C \bigg(\mathcal{F}(Q_N, \rho) + \|\omega - \omega_N\|_{L^2}^2 + N^{-\lambda} \\ & + \iint g(x-y)(\omega - \omega_N)(x)(\omega - \omega_N)(y) \,\mathrm{d}x \,\mathrm{d}y \bigg).\end{aligned} \tag{2.3.11}$$

We can bound R_4 as we did to obtain Inequality (2.3.5) (with $A = V$) and we get

$$|R_4| \leqslant C \|\nabla V\|_{L^\infty} \iint g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.3.12}$$

Finally we have

$$R_5 = \int \nabla g * u_N \,\mathrm{d}(\rho-\rho_N)$$

with $u_N = -\omega_N \nabla^\perp g * (\omega-\omega_N)$. Using Proposition 2.1.13 we get

$$\begin{aligned}|R_5| \leqslant& C\bigg(|\nabla g * u_N|_{\mathcal{C}^{0,\theta}} N^{-\lambda} + \left\|\nabla^2 g * u_N\right\|_{L^2} \bigg(\mathcal{F}(Q_N,\rho) \\ &+ (1+\|\rho\|_{L^\infty})N^{-1} + \frac{\ln(N)}{N}\bigg)^{\frac{1}{2}}\bigg)\end{aligned}$$

Using Morrey's inequality (see for example [14, Theorem 9.12]), Proposition 2.2.7 and Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]), we get that for some $p > 2$,

$$\begin{aligned}|\nabla g * u_N|_{\mathcal{C}^{0,\theta}} &\leqslant C\left\|\nabla^2 g * u_N\right\|_{L^p} \\ &\leqslant C\|u_N\|_{L^p} \\ &\leqslant C\|\omega_N\|_{L^\infty}\|\nabla g * (\omega-\omega_N)\|_{L^p} \\ &\leqslant C\|\omega_N\|_{L^\infty}\|\omega-\omega_N\|_{L^{\frac{2p}{p+2}}} \\ &\leqslant C\|\omega_N\|_{L^\infty}(\|\omega-\omega_N\|_{L^1}+\|\omega-\omega_N\|_{L^\infty}) \\ &\leqslant C\|\omega_{N,0}\|_{L^\infty}(\|\omega_0\|_{L^1}+\|\omega_{N,0}\|_{L^1}+\|\omega_0\|_{L^\infty}+\|\omega_{N,0}\|_{L^\infty})\end{aligned}$$

by Remark 2.1.3. Therefore by Assumption (2.1.7),

$$|\nabla g * u_N|_{\mathcal{C}^{0,\theta}} \leqslant C.$$

Now using Proposition 2.2.7 again, we get

$$\begin{aligned}\left\|\nabla^2 g * u_N\right\|_{L^2} &\leqslant C\|u_N\|_{L^2} \\ &\leqslant C\|\omega_N\|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

by Equation (2.3.1). Therefore by Assumption (2.1.7),

$$\begin{aligned}|R_5| \leqslant& C\bigg(\mathcal{F}(Q_N,\rho) + (1+\|\rho\|_{L^\infty})N^{-\lambda} \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\bigg)\end{aligned} \tag{2.3.13}$$

Combining inequalities (2.3.6), (2.3.10), (2.3.11), (2.3.12) and (2.3.13) we get

$$\left|\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t}\right| \leqslant C\Big(\mathcal{H}_N + \mathcal{F}(Q_N,\rho) + N^{-\beta}\Big).$$

for some $\beta > 0$. We are only remained to bound $\mathcal{F}(Q_N,\rho)$ by $\mathcal{H}_N$. Let us write

$$\begin{aligned}
\mathcal{F}(Q_N,\rho) = &\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)(\omega+\rho-\omega_N-\rho_N)^{\otimes 2}(\,\mathrm{d}x\,\mathrm{d}y)\\
&-\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)(\omega-\omega_N)(x)(\omega-\omega_N)(y)\,\mathrm{d}x\,\mathrm{d}y\\
&-2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y)\\
&\leqslant \mathcal{H}_N + 0 - 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} g(x-y)(\omega-\omega_N)(x)\,\mathrm{d}x\,\mathrm{d}(\rho-\rho_N)(y).
\end{aligned}$$

To bound the upper integral, we use Proposition 2.1.13 to get that

(2.3.14)

$$\begin{aligned}
&2\left|\int g*(\omega-\omega_N)\,\mathrm{d}(\rho-\rho_N)\right|\\
&\qquad\leqslant C\Bigg(|g*(\omega-\omega_N)|_{\mathcal{C}^{0,\theta}}N^{-\lambda} + 2C\,\|\nabla g*(\omega-\omega_N)\|_{L^2}\\
&\qquad\quad\times\left(\mathcal{F}(Q_N,\rho)+(1+\|\rho\|_{L^\infty})N^{-1}+\frac{\ln(N)}{N}\right)^{\frac{1}{2}}\Bigg)\\
&\qquad\leqslant C|g*(\omega-\omega_N)|_{\mathcal{C}^{0,\theta}}N^{-\lambda} + C\,\|\nabla g*(\omega-\omega_N)\|_{L^2}^2\\
&\qquad\quad+\frac{1}{2}\left(\mathcal{F}(Q_N,\rho)+(1+\|\rho\|_{L^\infty})N^{-1}+\frac{\ln(N)}{N}\right).
\end{aligned}$$

Now by Morrey's inequality (see for example [14, Theorem 9.12]) and Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]), for some $p > 2$,

(2.3.15)
$$\begin{aligned}
|g*(\omega-\omega_N)|_{\mathcal{C}^{0,\theta}} &\leqslant C\,\|\nabla g*(\omega-\omega_N)\|_{L^p}\\
&\leqslant C\,\|\omega-\omega_N\|_{L^{\frac{2p}{p+2}}}\\
&\leqslant C\,\|\omega_0\|_{L^{\frac{2p}{p+2}}} + \|\omega_{N,0}\|_{L^{\frac{2p}{p+2}}}\\
&\leqslant C(\|\omega_0\|_{L^1\cap L^\infty} + \|\omega_{N,0}\|_{L^1\cap L^\infty})\\
&\leqslant C
\end{aligned}$$

by Assumption (2.1.7). Using (2.3.1) we also have

$$\|\nabla g*(\omega-\omega_N)\|_{L^2}^2 \leqslant \mathcal{H}_N.$$

Therefore

$$\mathcal{F}(Q_N, \rho) \leqslant CN^{-\lambda} + C\mathcal{H}_N + \frac{1}{2}\mathcal{F}(Q_N, \rho)$$

for some for some $\lambda > 0$, hence

$$\mathcal{F}(Q_N, \rho) \leqslant C(\mathcal{H}_N + N^{-\lambda}). \tag{2.3.16}$$

It follows that

$$\frac{\mathrm{d}\mathcal{H}_N}{\mathrm{d}t} \leqslant C\Big(\mathcal{H}_N + N^{-\beta}\Big).$$

Applying Grönwall's lemma we get

$$\mathcal{H}_N(t) \leqslant (\mathcal{H}_N(0) + CN^{-\beta})e^{CT},$$

that is Inequality (2.1.10).

2.3.3 Proof of Proposition 2.1.9

Let φ be a smooth test function with compact support in $\mathbb{R}^4$. We have:

$$\begin{aligned}
&\int \varphi(x,\xi)\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{(q_i,p_i)} - \rho\otimes\delta_{\xi=v(x)}\right)(\,\mathrm{d}x\,\mathrm{d}\xi)\\
=&\frac{1}{N}\sum_{i=1}^{N}[\varphi(q_i,p_i) - \varphi(q_i,v(q_i))]\\
&+\frac{1}{N}\sum_{i=1}^{N}\varphi(q_i,v(q_i)) - \int\varphi(x,v(x))\rho(x)\,\mathrm{d}x\\
=:&\,T_1 + T_2.
\end{aligned}$$

Let us bound T_1:

$$\begin{aligned}
|T_1| &\leqslant \frac{1}{N}\sum_{i=1}^{N}\|\varphi(q_i,\cdot)\|_{W^{1,\infty}}|p_i - v(q_i)|\\
&\leqslant \|\varphi\|_{H^5}\frac{1}{N}\sum_{i=1}^{N}|p_i - v(q_i)|
\end{aligned}$$

by Sobolev embedding. Using Cauchy-Schwarz inequality we get

$$|T_1| \leqslant C\|\varphi\|_{W^{1,\infty}}\left(\frac{1}{N}\sum_{i=1}^{N}|p_i - v(q_i)|^2\right)^{\frac{1}{2}}. \tag{2.3.17}$$

For the second term:

$$|T_2| = \left| \int \varphi(x, v(x))(\rho - \rho_N)(\,\mathrm{d}x) \right| \leqslant \|\varphi \circ (I_d, v)\|_{H^2} \|\rho - \rho_N\|_{H^{-2}}.$$

Let $f := (I_d, v)$, then

$$\|\varphi \circ f\|_{H^2} \leqslant C(1 + \|\nabla v\|^2_{W^{1,\infty}}) \sup_{0 \leqslant k \leqslant 2} \left\| \nabla^k \varphi \circ f \right\|_{L^2}. \tag{2.3.18}$$

Now let $\psi := \partial^\alpha \varphi$ for some multi-index α of length $k \in \{0, 1, 2\}$, then

$$\begin{aligned} \|\psi \circ f\|^2_{L^2} &= \int |\psi(x, v(t, x))|^2 \,\mathrm{d}x \\ &\leqslant \sup_y \int |\psi(x, y)|^2 \,\mathrm{d}x \\ &\leqslant \|F\|_{W^{3,1}} \end{aligned}$$

where $F(y) := \int |\psi(x, y)|^2 \,\mathrm{d}x$ (see for example [14, Corollary 9.13]). Remark that

$$\begin{aligned} |\partial_{y_i} F(y)| &= 2 \left| \int \partial_{y_i} \psi(x, y) \psi(x, y) \,\mathrm{d}x \right| \\ &\leqslant \int |\partial_{y_i} \psi(x, y)|^2 \,\mathrm{d}x + \int |\psi(x, y)|^2 \,\mathrm{d}x. \end{aligned}$$

Doing the same computations for all derivatives of F of order less or equal than three and integrating in y gives us

$$\|F\|_{W^{3,1}} \leqslant C \|\psi\|^2_{H^3}.$$

From (2.3.18) and the upper equation we get

$$\|\varphi \circ f\|_{H^2} \leqslant C(1 + \|\nabla v\|^2_{W^{1,\infty}}) \|\varphi\|_{H^5}. \tag{2.3.19}$$

Now, by [75, Proposition 3.10] (which is a refined version of Proposition 2.1.13), we have

$$\|\rho - \rho_N\|_{H^{-2}} \leqslant C(|\mathcal{F}(Q_N, \rho)|^{\frac{1}{2}} + N^{-\frac{1}{2}} |\ln(N)|^{\frac{1}{2}} + (1 + \|\rho\|_{L^\infty}) N^{-\frac{1}{2}}).$$

Using Assumption (2.1.11) we can bound $\mathcal{F}(Q_N, \rho)$ as in (2.3.16) to get

$$|\mathcal{F}(Q_N, \rho)| \leqslant C(\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N) + N^{-\lambda}) \tag{2.3.20}$$

and therefore

$$\|\rho - \rho_N\|_{H^{-2}} \leqslant C(\mathcal{H}(\omega, \rho, v, \omega_N, Q_N, P_N)^{\frac{1}{2}} + (1 + \|\rho\|_{L^\infty}) N^{-\lambda})$$

for some $\lambda > 0$. Combining the upper inequality with (2.3.17) and (2.3.19) we get that

$$\left|\int \varphi(x,\xi)\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{(q_i,p_i)} - \rho\otimes\delta_{\xi=v(x)}\right)\right|$$
$$\leqslant C(1+\|\nabla v\|_{W^{1,\infty}}^2)\,\|\varphi\|_{H^5}\left(\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N)^{\frac{1}{2}}+C(1+\|\rho\|_{L^\infty})N^{-\beta}\right)^{\frac{1}{2}}$$

for some $\beta > 0$. Thus we get (2.1.12). It follows from this estimate that if

$$\mathcal{H}(\omega,\rho,v,\omega_N,Q_N,P_N) \underset{N\to\infty}{\longrightarrow} 0$$

then

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{(q_i,p_i)} \underset{H^{-5}}{\longrightarrow} \rho\otimes\delta_{\xi=v(t,x)}.$$

By equality (2.3.1) we also have

$$\|\nabla g*(\omega_N-\omega)\|_{L^2}^2 + \|\omega-\omega_N\|_{L^2}^2 \underset{N\to+\infty}{\longrightarrow} 0$$

Now remark that for any $\mu \in L^2$,

$$\begin{aligned}\|\nabla g*\mu\|_{L^2}^2 &= C\left\|\widehat{\nabla g}\widehat{\mu}\right\|_{L^2}^2\\ &= C\int\frac{|\widehat{\mu}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi \qquad (2.3.21)\\ &= C\,\|\mu\|_{\dot{H}^{-1}}^2\end{aligned}$$

and therefore

$$\omega_N-\omega \underset{N\to+\infty}{\longrightarrow} 0 \text{ in } L^2\cap\dot{H}^{-1}.$$

Finally, using Inequality (2.3.20), we have

$$\mathcal{F}(Q_N,\rho) \underset{N\to+\infty}{\longrightarrow} 0$$

and therefore by [75, Proposition 3.10] we get that for any $a < -1$,

$$\rho_N \underset{N\to+\infty}{\longrightarrow} \rho \text{ in } H^a$$

which concludes the proof of Proposition 2.1.9.

2.3.4 Proof of Proposition 2.1.12

We have

$$\begin{aligned}\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} & g(x-y)(\rho_0+\omega_0-\rho_{N,0}-\omega_{N,0})^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y)\\ &= \mathcal{F}(Q_{N,0},\rho_0)+\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y)\\ &\quad -2\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})(x)\,\mathrm{d}x\,\mathrm{d}(\rho_0-\rho_{N,0})(y)\end{aligned}$$

It is proved in Theorem 1.1 of [26] that the weak-$*$ convergence of $\rho_{N,0}$ to ρ_0 and the convergence of

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g(q_i^0-q_j^0)$$

to

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\rho_0(x)\rho_0(y)\,\mathrm{d}x\,\mathrm{d}y$$

ensures that

$$\mathcal{F}(Q_{N,0},\rho_0)\underset{N\to+\infty}{\longrightarrow}0. \tag{2.3.22}$$

Using (2.3.1), (2.3.21) and the convergence of $\omega_{N,0}$ to ω_0 in $\dot{H}^{-1}$ we have that

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})^{\otimes 2}(\mathrm{d}x\,\mathrm{d}y)\underset{N\to+\infty}{\longrightarrow}0.$$

Using inequalities (2.3.14) and (2.3.15) we have

$$\begin{aligned}&\left|\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})(x)\,\mathrm{d}x\,\mathrm{d}(\rho_0-\rho_{N,0})(y)\right|\\ &\leqslant C\Bigg((\|\omega_0\|_{L^1\cap L^\infty}+\|\omega_{N,0}\|_{L^1\cap L^\infty})N^{-\lambda}\\ &\quad+\|\nabla g*(\omega_0-\omega_{N,0})\|_{L^2}\left(\mathcal{F}(Q_{N,0},\rho_0)+(1+\|\rho_0\|_{L^\infty})N^{-1}+\frac{\ln(N)}{N}\right)^{\frac{1}{2}}\Bigg).\end{aligned}$$

Using Assumption (2.1.13), equations (2.3.1), (2.3.21), (2.3.22) and the convergence of $\omega_{N,0}$ to ω_0 in $\dot{H}^{-1}$ we get that

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g(x-y)(\omega_0-\omega_{N,0})(x)\,\mathrm{d}x\,\mathrm{d}(\rho_0-\rho_{N,0})(y)\underset{N\to+\infty}{\longrightarrow}0$$

which ends the proof of Proposition 2.1.12.

Chapitre 3

Mean-field limit of a system of point vortices for the lake equations

The results of this chapter are the content of [68].

3.1 Introduction

The purpose of this chapter is to investigate the mean-field limit of the system of introduced in Section 1.1.4 modeling the evolution of a finite number of small vortices in a lake:

$$\dot{q}_i = -\alpha_N \frac{\nabla^\perp b(q_i)}{b(q_i)} - \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{b(q_i)} \nabla_x^\perp g_b(q_i, q_j) \tag{3.1.1}$$

where

— $\alpha_N := \frac{|\ln(\varepsilon_N)|}{4\pi N}$ where ε_N is the size of the vortices.

— $\perp$ denotes the rotation by $\frac{\pi}{2}$ (that is $(x_1, x_2)^\perp := (-x_2, x_1)$).

— $b : \mathbb{R}^2 \longrightarrow (0, +\infty)$ is the depth function satisfying Assumption 3.1.3 below.

In the regime where $\alpha_N \underset{N\to+\infty}{\longrightarrow} \alpha \in [0, +\infty)$ we want to recover the forced lake equations (introduced in Section 1.1.4):

$$\begin{cases} \partial_t \omega + \operatorname{div}\left(\left(u - \alpha \frac{\nabla^\perp b}{b}\right)\omega\right) = 0 \\ \operatorname{div}(bu) = 0 \\ \operatorname{curl}(u) = \omega \end{cases} \tag{3.1.2}$$

where

— $\alpha \in [0, +\infty)$ is a forcing parameter.
— $u : [0, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the velocity field of the fluid.
— $\omega : [0, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the vorticity field of the fluid, defined by

$$\omega = \mathrm{curl}(u) := \partial_1 u_2 - \partial_2 u_1.$$

We will consider the following definition of weak solutions:

Definition 3.1.1. *Let $T > 0$ and $\omega_0 \in L^\infty(\mathbb{R}^2)$ with compact support. We say that (ω, u) is a weak solution of (3.1.2) on $[0, T]$ with initial condition ω_0 if $\omega \in L^1([0,T], L^\infty(\mathbb{R}^2, \mathbb{R}^2)) \cap \mathcal{C}^0([0,T], L^\infty(\mathbb{R}^2) - w^*)$ with compact support in space for all $t \in [0,T]$, $u \in L^2_{\mathrm{loc}}([0,T] \times \mathbb{R}^2, \mathbb{R}^2)$, for almost every $t \in [0,T]$, $\mathrm{div}(bu) = 0$ and $\mathrm{curl}(u) = \omega$ distributionally and for all φ smooth with compact support in $[0,T) \times \mathbb{R}^2$ and $t \in [0,T)$,*

$$\iint_{[0,t]\times\mathbb{R}^2} \partial_t \varphi \omega + \nabla\varphi \cdot \left(u - \alpha \frac{\nabla^\perp b}{b} \right) \omega = \int_{\mathbb{R}^2} \varphi(t)\omega(t) - \int_{\mathbb{R}^2} \varphi(0)\omega_0. \tag{3.1.3}$$

Two quantities will be of interest for the study of this limit. The interaction energy

$$E_N(t) := \frac{1}{N^2} \sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N g_b(q_i(t), q_j(t))$$

and the moment of inertia

$$I_N(t) := \frac{1}{N} \sum_{i=1}^N |q_i(t)|^2.$$

One could prove that the total energy

$$E_N^{tot} := E_N + \frac{\alpha_N}{N} \sum_{i=1}^N b(q_i)$$

is a conserved quantity for the point vortex system (3.1.1) or that if ω is a solution of (3.1.2) with enough regularity and decay, the quantity

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(t,x)\omega(t,y)\,\mathrm{d}x\,\mathrm{d}y + \alpha \int_{\mathbb{R}^2} b(x)\omega(t,x)\,\mathrm{d}x$$

is conserved by the flow. The moment of inertia I_N and the interaction energy E_N are not conserved quantity but they are bounded in time, and this will be useful both for our mean-field limit result and for the well-posedness of System (3.1.1) (see Section 3.3).

In the regime where the self-interaction of the point vortices are predominant ($\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$), we will study the mean-field limit of System (3.1.1) in an accelerated timescale:

$$\overline{q_i}(t) := q_i(\alpha_N^{-1}t).$$

This gives

$$\dot{\overline{q_i}} = -\frac{\nabla^\perp b(\overline{q_i})}{b(\overline{q_i})} - \frac{1}{N\alpha_N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{b(\overline{q_i})}\nabla_x^\perp g_b(\overline{q_i},\overline{q_j}). \tag{3.1.4}$$

In this regime the system of point vortices will converge to a transport equation along the level sets of the topography:

$$\partial_t\overline{\omega} - \operatorname{div}\left(\frac{\nabla^\perp b}{b}\overline{\omega}\right) = 0. \tag{3.1.5}$$

For this equation we will use the following definition of weak solutions:

Definition 3.1.2. *Let $T > 0$ and $\overline{\omega}_0 \in L^\infty(\mathbb{R}^2)$ with compact support. We say that is a weak solution of (3.1.5) on $[0,T]$ with initial condition $\overline{\omega}_0$ if $\overline{\omega} \in L^1([0,T], L^\infty(\mathbb{R}^2,\mathbb{R}^2)) \cap \mathcal{C}^0([0,T], L^\infty(\mathbb{R}^2) - w^*)$ with compact support in space for all $t \in [0,T]$ and for all φ smooth with compact support in $[0,T) \times \mathbb{R}^2$ and $t \in [0,T)$,*

$$\iint_{[0,t]\times\mathbb{R}^2} \partial_t\varphi\overline{\omega} - \nabla\varphi\cdot\frac{\nabla^\perp b}{b}\overline{\omega} = \int_{\mathbb{R}^2}\varphi(t)\overline{\omega}(t) - \int_{\mathbb{R}^2}\varphi(0)\overline{\omega}_0. \tag{3.1.6}$$

We also define the rescaled interaction energy

$$\overline{E_N}(t) := E_N(\alpha_N^{-1}t)$$

and the rescaled moment of inertia

$$\overline{I_N}(t) := I_N(\alpha_N^{-1}t).$$

3.1.1 Notations and assumptions

Notations

- For $u \in L^1_{\text{loc}}(\mathbb{R}^2,\mathbb{R}^2)$, we denote $\operatorname{curl}(u) = \partial_1 u_2 - \partial_2 u_1$.
- For $h \in \dot{H}^1(\mathbb{R}^2)$, we denote

$$[h,h]_{i,j} := 2\partial_i h\partial_j h - |\nabla h|^2\delta_{i,j}. \tag{3.1.7}$$

It is the stress-energy tensor used in [81] to prove the mean-field limit of several singular ODE's. Remark that for h smooth enough, we have

$$\operatorname{div}[h,h] = 2\Delta h\nabla h.$$

— We denote $<x>= (1+|x|^2)^{\frac{1}{2}}$.
— g is the opposite of the Green function of the laplacian:

$$g(x) := -\frac{1}{2\pi}\ln|x|.$$

— $|\cdot|_{\mathcal{C}^{0,s}}$ is the semi-norm associated to the Hölder space $\mathcal{C}^{0,s}$:

$$|f|_{\mathcal{C}^{0,s}} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^s}.$$

— When $1 \leqslant p \leqslant +\infty$, p' denotes the dual exponent of p.
— If ν is a probability measure on $\mathbb{R}^2$, we will denote $\nu^{\otimes 2} := \nu\otimes\nu$.
— C is a generic constant. We will denote $C_{A,B}$ when a constant depends on some quantities A and B.
— $\mathcal{P}(\mathbb{R}^2)$ is the space of probability measures on $\mathbb{R}^2$.
— For $Q_N = (q_1, ..., q_N) \in (\mathbb{R}^2)^N$ we denote $I(Q_N) = \dfrac{1}{N}\displaystyle\sum_{i=1}^N |q_i|^2$.

Assumptions

We will make the following assumption on the depth function b:

Assumption 3.1.3. *We assume that b is a smooth function, $\inf b > 0$, $\sup b < +\infty$ and that there exists $\gamma > 0$ such that*

$$(1+|x|)^{4+\gamma}(|\nabla b(x)| + |D^2 b(x)|) < +\infty.$$

We will consider regular solutions of (3.1.2) and (3.1.5) in the following sense:

Assumption 3.1.4. *We say that a function $\omega(t,x)$ satisfies Assumption 3.1.4 if $\omega \in L^\infty([0,T], L^\infty(\mathbb{R}^2)\cap\mathcal{P}(\mathbb{R}^2))\cap\mathcal{C}^0([0,T], L^\infty(\mathbb{R}^2) - w^*)$, if there exists a compact K such that for every $t\in[0,T]$, $\mathrm{supp}(\omega(t))\subset K$ and if $\nabla G_b[\omega] \in L^\infty([0,T], W^{1,\infty})$ where G_b is the operator defined by Equation (3.2.11).*

Remark 3.1.5. A weak solution of (3.1.2) in the sense of Definition 3.1.1 (or a weak solution of (3.1.5) in the sense of Definition 3.1.2) does not necessarily verify Assumption 3.1.4 because of the regularity we ask for the velocity field $\nabla G_b[\omega]$. This assumption will be crucial to apply Proposition 3.6.1 and prove the mean-field limit Theorem 3.1.6. The existence and uniqueness of sufficiently regular solutions of (3.1.2) locally in time is ensured by [27, Theorem 2]. One could also prove that $\omega\in L^\infty([0,T],\mathcal{C}^{0,s})$ is sufficient to have $\nabla G_b[\omega]\in L^\infty([0,T], W^{1,\infty})$.

3.1.2 Main result and plan of the chapter

The main result of this paper is the following theorem which gives the mean-field limit of the point vortex system (3.1.1) and its rescaled version (3.1.4) (we recall that the kernel g_b is defined by (3.2.10)):

Theorem 3.1.6. *Assume that b satisfies Assumption 3.1.3. We have mean-field convergence of the point-vortex system in the two following regimes:*

1. *Let ω be a solution of (3.1.2) with initial datum ω_0 in the sense of Definition 3.1.1, satisfying Assumption 3.1.4 and $(q_1, ..., q_N)$ be a solution of (3.1.1). Assume that:*
   — *$(I_N(0))_N$ is bounded.*
   — $\displaystyle\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i^0} \xrightarrow[N\to+\infty]{*} \omega_0$ *for the weak-$*$ topology of probability measures.*
   — $\alpha_N \underset{N\to+\infty}{\longrightarrow} \alpha$.
   — $\displaystyle\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i^0, q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_0(x)\omega_0(y)\,\mathrm{d}x\,\mathrm{d}y.$

   Then for all $t\in[0,T]$, $\displaystyle\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i(t)} \xrightarrow[N\to+\infty]{*} \omega(t)$ for the weak-$*$ topology of probability measures and

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i(t), q_j(t)) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(t,x)\omega(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

2. *Let $\overline{\omega}$ be a solution of (3.1.5) with initial datum ω_0 in the sense of Definition 3.1.2, satisfying Assumption 3.1.4 and $(\overline{q_1}, ..., \overline{q_N})$ be a solution of (3.1.4). Assume that:*
   — *$(\overline{I_N}(0))_N$ is bounded.*
   — $\displaystyle\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i^0} \xrightarrow[N\to+\infty]{*} \omega_0$ *for the weak-$*$ topology of probability measures.*
   — $\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$.
   — $\displaystyle\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i^0, q_j^0) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_0(x)\omega_0(y)\,\mathrm{d}x\,\mathrm{d}y.$

   Then for all $t\in[0,T]$, $\displaystyle\frac{1}{N}\sum_{i=1}^{N}\delta_{\overline{q_i}(t)} \xrightarrow[N\to+\infty]{*} \overline{\omega}(t)$ for the weak-$*$ topology of probability measures and

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(\overline{q_i}(t), \overline{q_j}(t)) \underset{N\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\overline{\omega}(t,x)\overline{\omega}(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

As explained in Section 1.3.2 the proof of this Theorem will rely on the following "modulated energy" functionnal. For an empirical measure of point vortices $(q_1, ..., q_N)$ and a vorticity field $\omega \in L^\infty$ with compact support, it is defined by:

$$(3.1.8)\quad \mathcal{F}_b(Q_N, \omega) := \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right)(x)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right)(y)$$

where

$$\Delta := \{(x,x)\ ;\ x \in \mathbb{R}^2\}.$$

We will use this energy to control the distance between solutions ω and Q_N of (3.1.2) and (3.1.1) or solutions $\overline{\omega}$ and $\overline{Q_N}$ of (3.1.5) and (3.1.4) at any given time t:

$$(3.1.9)\qquad \mathcal{F}_{b,N}(t) := \mathcal{F}_b(Q_N(t), \omega(t))$$

and

$$(3.1.10)\qquad \overline{\mathcal{F}}_{b,N}(t) := \mathcal{F}_b(\overline{Q_N}(t), \overline{\omega}(t)).$$

The proof of Theorem 3.1.6 relies on Grönwall-type estimates on these two quantities. The paper is organised as follows:

— In Section 3.2 we prove the well-posedness of the elliptic problem linking a velocity field satisfying $\mathrm{div}(bu) = 0$ and its vorticity, the existence of a Green kernel for this elliptic problem and we establish several regularity estimates.

— In Section 3.3 we prove that the point-vortex system is well-posed and give some estimates on the interaction energy and on the moment of inertia of the system that we will need in Section 3.7.

— In Section 3.4 we compute the time derivative of $\mathcal{F}_{b,N}$ and of $\overline{\mathcal{F}}_{b,N}$.

— In Section 3.5 we state several properties of the modulated energy. We prove that it controls the convergence in H^s for $s < -1$ (see Corollary 3.5.3) and that having the convergence of the modulated energy is equivalent to have weak-$*$ convergence of the point vortex system and convergence of its interaction energy (see Corollary 3.5.4).

— In Section 3.6 we bound the main term appearing in the derivatives of the modulated energies.

— In Section 3.7 we use the results of the other sections to prove Theorem 3.1.6.

The modulated energy $\mathcal{F}_b$ is similar to the modulated energy defined by Equation (1.2.8) and the proofs of Sections 3.4 to 3.7 follow the same scheme as the one we explained in Section 1.2.2. The main difference between Theorem 3.1.6 and the classical mean-field result on the Coulomb kernel (see

Theorem 1.2.1) is that the kernel g_b is not of the form $a(x, y) = a(x - y)$. Most of the difficulties adressed by this paper consist in dealing with the heterogeneity of the kernel g_b.

3.2 Velocity reconstruction

There exists a Biot-Savart type law to reconstruct a velocity field u satisfying $\operatorname{div}(bu) = 0$ from its vorticity. In this section we prove several results concerning this reconstruction. In Subsection 3.2.1 we prove that the elliptic equations linking u with its vorticity are well-posed. In Subsection 3.2.2 we prove some results related to the asymptotic behavior of the velocity field as $|x| \longrightarrow \infty$. In Subsection 3.2.3, we give an analogue of the Biot-Savart law for a velocity field satisfying System (3.2.1). Finally, in Subsection 3.2.4 we define some regularisations of the Coulomb kernel and of the dirac mass that we will need in Sections 3.5 and 3.6.

3.2.1 Well-posedness of the elliptic problem

In this subsection we justify the well-posedness of the elliptic equations satisfied by the velocity field:

$$\begin{cases} \operatorname{div}(bu) = 0 \\ \operatorname{curl}(u) = \omega. \end{cases} \tag{3.2.1}$$

As we will write $u = -\frac{1}{b}\nabla^{\perp}\psi$ we will also consider the "stream function" formulation of the upper system:

$$-\operatorname{div}\left(\frac{1}{b}\nabla\psi\right) = \omega. \tag{3.2.2}$$

For this purpose we will consider the following weighted Sobolev spaces:

Definition 3.2.1. *For $1 < p < \infty$ we consider the Banach space $W^{2,p}_{-1}(\mathbb{R}^2)$ defined by*

$$W^{2,p}_{-1}(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2)\ ;\ \forall \alpha \in \mathbb{N}^2, |\alpha| \leqslant 2, <\cdot>^{|\alpha|-1} D^{\alpha} u \in L^p(\mathbb{R}^2)\}$$

and equipped with its natural norm

$$\|u\|_{W^{2,p}_{-1}} := \left(\sum_{|\alpha|\leqslant 2} \left\| <\cdot>^{|\alpha|-1} D^{\alpha} u \right\|^p_{L^p}\right)^{\frac{1}{p}}.$$

These weighted spaces were first introduced by Cantor in [15] and have been investigated to study elliptic equations on unbounded domains. For a more precise study of these spaces and further references we refer to [59, 65, 66]. The following proposition is a straightforward consequence of [59, Theorem 2] (which is the combination of two theorems proved in [65] and [66]) and states that Equations (3.2.1) and (3.2.2) are well-posed.

Proposition 3.2.2. *Let $2 < p < +\infty$, assume that $<\cdot>\omega \in L^p(\mathbb{R}^2)$, then there exists a unique solution ψ of* (3.2.2) *in $W^{2,p}_{-1}(\mathbb{R}^2)/\mathbb{R}$. Morever if $u \in L^p(\mathbb{R}^2, \mathbb{R}^2)$ is a solution of* (3.2.1) *in the sense of distributions, then*

$$u = -\frac{1}{b}\nabla^{\perp}\psi.$$

Proof. We can rewrite Equation (3.2.2) as

$$-\Delta\psi - b\nabla\left(\frac{1}{b}\right)\cdot\nabla\psi = b\omega.$$

We have that:

- $-\Delta$ is an elliptic operator with constant coefficients and homogeneous of degree 2.
- $b\nabla\left(\frac{1}{b}\right) \in \mathcal{C}^0$ and

  $$\lim_{|x|\to+\infty}\left|<x>^{2-1+0} b(x)\nabla\left(\frac{1}{b}\right)(x)\right| = 0$$

  since b satisfies Assumption 3.1.3.
- $<\cdot>b\omega \in L^p$.
- $-1 \leqslant -\frac{2}{p}$ and $1 - \frac{2}{p} \notin \mathbb{N}$.

Therefore by [59, Theorem 2], there exists a unique solution ψ (up to a constant) of Equation (3.2.2) in $W^{2,p}_{-1}(\mathbb{R}^2)$.

Now if $u \in L^p$ is a solution of (3.2.1), then

$$\begin{aligned}\|<\cdot>\operatorname{curl}(bu)\|_{L^p} &= \|<\cdot>b\omega\|_{L^p} + \left\|<\cdot>\nabla^{\perp}b\cdot u\right\| \\ &\leqslant C_b(\|<\cdot>\omega\|_{L^p} + \|u\|_{L^p})\end{aligned}$$

since b satisfies Assumption 3.1.3. Let us consider $\pi \in W^{2,p}_{-1}(\mathbb{R}^2)$ to be the unique solution (up to a constant) of $-\Delta\pi = \operatorname{curl}(bu)$ given by [59, Theorem 1]. Then $bu + \nabla^{\perp}\pi$ is a div-curl free vector field in L^p so it is zero. Moreover,

$$-\operatorname{div}\left(\frac{1}{b}\nabla\pi\right) = -\operatorname{curl}\left(\frac{1}{b}\nabla^{\perp}\pi\right) = \operatorname{curl}(u) = \omega$$

so $\nabla\pi = \nabla\psi$ by uniqueness of solutions of (3.2.2) in $W^{2,p}_{-1}(\mathbb{R}^2)/\mathbb{R}$. □

Now we state several estimates for solutions of Equation (3.2.2), proved by Duerinckx in [27]:

Lemma 3.2.3. *[From [27, Lemma 2.6]] Let $p > 2$, ω be such that $<\cdot> \omega \in L^p(\mathbb{R}^2)$. If $\psi \in W^{2,p}_{-1}(\mathbb{R}^2)$ is the solution of* (3.2.2) *given by Proposition 3.2.2, then:*

1. *There exists $p_0 > 2$ depending only on b such that for all $2 < p \leqslant p_0$,*
$$\|\nabla\psi\|_{L^p} \leqslant C_p \|\omega\|_{L^{\frac{2p}{p+2}}} .$$
2. *For all $0 < s < 1$,*
$$|\nabla\psi|_{\mathcal{C}^{0,s}} \leqslant C_s \|\omega\|_{L^{\frac{2}{1-s}}} .$$
3. $\|\nabla\psi\|_{L^\infty} \leqslant C \|\omega\|_{L^1 \cap L^\infty}$.

Remark 3.2.4. In [27], this lemma was stated for any solution of (3.2.2) with decreasing gradient (which is the case for a solution given by Proposition 3.2.2 since its gradient is in $W^{1,p}$) and for ω smooth with compact support but by density it can be extended to all ω such that $<\cdot> \omega \in L^p(\mathbb{R}^2)$ and such that the upper inequalities make sense.

3.2.2 Asymptotic behavior of the velocity field

The main result of this subsection is the following proposition giving the asymptotic behavior of a velocity field satisfying (3.2.1).

Proposition 3.2.5. *Let $\omega \in L^\infty$ with compact support and $u = -\frac{1}{b}\nabla^\perp\psi$ where ψ is the solution of* (3.2.2) *given by Proposition 3.2.2. There exists $C > 0$ depending only on b and ω such that for all $x \in \mathbb{R}^2\backslash\{0\}$,*

$$\left| u(x) - \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} \omega\right)\frac{x^\perp}{|x|^2}\right| \leqslant \frac{C}{|x|^2}. \tag{3.2.3}$$

Moreover there exists $\delta \in (0,1)$ and C such that

$$|\psi(x)| \leqslant C(1 + |x|^\delta). \tag{3.2.4}$$

To prove this proposition we will need to use the following result about the asymptotic behavior of a velocity field given by the usual Biot-Savart law:

Lemma 3.2.6. *Let us assume that μ is a measurable function such that $\mu \in L^1((1+|x|^2)\,\mathrm{d}x)$ and $|\cdot|^2\mu \in L^p$ for some $p > 2$. Then there exists $C, R > 0$ depending only on μ such that for all $x \in \mathbb{R}^2\backslash\{0\}$,*

$$\left| \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \,\mathrm{d}\mu(y) - \left(\int_{\mathbb{R}^2} \mathrm{d}\mu(y) \right) \frac{x}{|x|^2} \right| \leqslant \frac{C}{|x|^2}.$$

In particular if $\int_{\mathbb{R}^2} \mathrm{d}\mu = 0$, *then*

$$\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \,\mathrm{d}\mu(y) = \underset{|x|\to+\infty}{\mathcal{O}} (|x|^{-2}).$$

This lemma is a classical result in fluid dynamics (see for example [61, Proposition 3.3]) that we will prove for the sake of completeness.

Proof. If $x \neq 0$, we have

$$\int_{\mathbb{R}^2} \mu(y) \left(\frac{x-y}{|x-y|^2} - \frac{x}{|x|^2} \right) \mathrm{d}y = \frac{1}{|x|^2} \int_{\mathbb{R}^2} \mu(y) \frac{|x|^2(x-y) - x|x-y|^2}{|x-y|^2} \,\mathrm{d}y.$$

Now remark that

$$\begin{aligned} &|x|^2(x-y) - x|x-y|^2 \\ &\quad = |x|^2(x-y) - (x-y)(|x|^2 + |y|^2 - 2x\cdot y) - y|x-y|^2 \\ &\quad = (x-y)(|y|^2 - 2(x-y)\cdot y - 2|y|^2) - y|x-y|^2 \\ &\quad = -y|x-y|^2 - 2[(x-y)\cdot y](x-y) - |y|^2(x-y). \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \mu(y) \left(\frac{x-y}{|x-y|^2} - \frac{x}{|x|^2} \right) \mathrm{d}y \right| \leqslant & \frac{C}{|x|^2} \left(\int_{\mathbb{R}^2} |y||\mu(y)| \,\mathrm{d}y \right. \\ & \left. + \int_{\mathbb{R}^2} \frac{|y|^2|\mu(y)|}{|x-y|} \,\mathrm{d}y \right). \end{aligned}$$

Now we have that for any $p > 2$,

$$\int_{\mathbb{R}^2} \frac{|y|^2|\mu(y)|}{|x-y|} \,\mathrm{d}y \leqslant \| |\cdot|^2 \mu \|_{L^1}^{\frac{p-2}{2p-2}} \| |\cdot|^2 \mu \|_{L^p}^{\frac{p}{2p-2}}$$

(see for example [46, Lemma 1]) and therefore we get the proof of Lemma 3.2.6. □

With this result we can now study the asymptotic behavior of a velocity field satisfying System (3.2.1):

Proof of Proposition 3.2.5. We write

$$\mu := \operatorname{div}(u) = \operatorname{div}\left(\frac{1}{b} bu \right) = \nabla \left(\frac{1}{b} \right) \cdot bu = -\frac{\nabla b \cdot u}{b}. \tag{3.2.5}$$

By Helmholtz decomposition we can write

$$u = -\nabla g * \mu - \nabla^{\perp} g * \omega. \tag{3.2.6}$$

Let $2 < p < +\infty$, then by Assumption 3.1.3,

$$\begin{aligned}\int_{\mathbb{R}^2} (1+|y|^2)|\mu(y)|\,\mathrm{d}y &\leqslant C_b \int_{\mathbb{R}^2} \frac{1+|y|^2}{(1+|y|)^{4+\gamma}}|u(y)|\,\mathrm{d}y \\ &\leqslant C_b \left\|(1+|\cdot|)^{-(2+\gamma)}\right\|_{L^{p'}} \|u\|_{L^p} < +\infty\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} |y|^{2p}|\mu(y)|^p\,\mathrm{d}y &\leqslant C_b \int_{\mathbb{R}^2} |y|^{2p}(1+|y|)^{-p(4+\gamma)}|u(y)|^p\,\mathrm{d}y \\ &\leqslant C_b \int_{\mathbb{R}^2} |u(y)|^p\,\mathrm{d}y < +\infty.\end{aligned}$$

If we apply Lemma 3.2.6 on each term of (3.2.6) we only need to show that $\displaystyle\int \mu = 0$ to obtain (3.2.3). We define

$$b_\infty := \lim_{|x|\to+\infty} b(x).$$

Remark that the existence of this limit is guaranteed by Assumption 3.1.3. Let us prove by induction that for any integer n,

$$\sum_{k=0}^{n} \frac{\ln^k(b_\infty)}{k!} \int_{\mathbb{R}^2} \mu = \frac{1}{n!} \int_{\mathbb{R}^2} \ln^n(b)\mu. \tag{3.2.7}$$

If $n = 0$ then this equality reduces to $\displaystyle\int \mu = \int \mu$. Now let us assume that it holds for some $n \geqslant 0$. Using Equation (3.2.5), we get

$$\ln^n(b)\mu = -\frac{1}{n+1}\nabla \ln^{n+1}(b)\cdot u.$$

Inserting Equation (3.2.6), we get

$$\ln^n(b)\mu = \frac{1}{n+1}\nabla \ln^{n+1}(b)\cdot(\nabla g * \mu + \nabla^{\perp} g * \omega).$$

Integrating over a ball of center 0 and radius R and integrating by parts we get

$$\begin{aligned}\int_{B(0,R)} \ln^n(b)\mu =& \frac{1}{n+1}\left(\int_{\partial B(0,R)} \ln^{n+1}(b)(\nabla g * \mu + \nabla^\perp g * \omega)\cdot \mathrm{d}\vec{S}\right.\\ &\left. - \int_{B(0,R)} \ln^{n+1}(b)\,\mathrm{div}(\nabla g * \mu + \nabla^\perp g * \omega)\right)\\ =&\frac{1}{n+1}\left(\int_{\partial B(0,R)} \ln^{n+1}(b)(\nabla g * \mu + \nabla^\perp g * \omega)\cdot \mathrm{d}\vec{S}\right.\\ &\left. + \int_{B(0,R)} \ln^{n+1}(b)\mu\right)\end{aligned} \tag{3.2.8}$$

where $\mathrm{d}\vec{S}(x) = 2\pi x\,\mathrm{d}\sigma(x)$ and σ is the uniform probability measure on $\partial B(0,R)$. Using Lemma 3.2.6, we get that for $x \in \partial B(0,R)$,

$$\begin{aligned}(\nabla g &* \mu + \nabla^\perp g * \omega)(x)\cdot x\\ &= -\frac{1}{2\pi}\left(\left(\int_{\mathbb{R}^2}\mu\right)\frac{x}{|x|^2} + \left(\int_{\mathbb{R}^2}\omega\right)\frac{x^\perp}{|x|^2} + \mathcal{O}(R^{-2})\right)\cdot x\\ &= -\frac{1}{2\pi}\left(\int_{\mathbb{R}^2}\mu\right) + \mathcal{O}(R^{-1}).\end{aligned}$$

Thus we get that

$$\frac{1}{n+1}\int_{\partial B(0,R)} \ln^{n+1}(b)(\nabla g * \mu + \nabla^\perp g * \omega)\cdot \mathrm{d}\vec{S} \underset{R\to+\infty}{\longrightarrow} -\frac{\ln^{n+1}(b_\infty)}{n+1}\int_{\mathbb{R}^2}\mu.$$

Combining the upper equality with Equations (3.2.7) and (3.2.8) we get that

$$\sum_{k=0}^{n+1}\frac{\ln^k(b_\infty)}{k!}\int_{\mathbb{R}^2}\mu = \frac{1}{(n+1)!}\int_{\mathbb{R}^2}\ln^{n+1}(b)\mu$$

which ends the proof of Equality (3.2.7). Now if n goes to infinity, this gives

$$e^{\ln(b_\infty)}\int_{\mathbb{R}^2}\mu = 0$$

and thus

$$\int_{\mathbb{R}^2}\mu = 0.$$

Now by Lemma 3.2.3 and Morrey's inequality (see for example [14, Theorem 9.12]), for any $2 < p \leqslant p_0$,

$$\begin{aligned}|\psi(x)| &\leqslant |\psi(x) - \psi(0)| + |\psi(0)|\\ &\leqslant C_p\,\|\nabla\psi\|_{L^p}\,|x|^{1-\frac{2}{p}} + |\psi(0)|.\end{aligned}$$

Taking $\delta = 1 - \dfrac{2}{p}$ we obtain (3.2.4). $\square$

3.2.3 Construction of the Green kernel

The main result of this subsection is a Biot-Savart type law for the lake equations, given by Proposition 3.2.8. Let us begin by giving the definition and some estimates on the function S_b that appears in the definition of the kernel g_b (see Equation (3.2.10)):

Lemma 3.2.7. *For $y \in \mathbb{R}^2$, let $S_b(\cdot, y)$ be a solution of*

$$-\operatorname{div}\left(\frac{1}{b}\nabla S_b(\cdot, y)\right) = -g(\cdot - y)\sqrt{b(y)}\Delta\left(\frac{1}{\sqrt{b}}\right) \tag{3.2.9}$$

given by Proposition 3.2.2 applied to $\omega = -g(\cdot - y)\sqrt{b(y)}\Delta\left(\frac{1}{\sqrt{b}}\right)$ and $\psi = S_b(\cdot, y)$. Then:

1. *For any $y \in \mathbb{R}^2$ and $2 < p \leqslant +\infty$, $\nabla_x S_b(\cdot, y) \in L^p$ and*
$$\|\nabla_x S_b(\cdot, y)\|_{L^p} \leqslant C_{b,p}(1 + |y|).$$
2. *There exists $s_0 \in (0, 1)$ such that for all $0 < s < s_0$,*
$$|\nabla_x S_b(x, \cdot)|_{\mathcal{C}^{0,s}(B(y,1))} \leqslant C_{b,s}(1 + |y|)$$
$$|\nabla_x S_b(\cdot, y)|_{\mathcal{C}^{0,s}(\mathbb{R}^2)} \leqslant C_{b,s}(1 + |y|).$$

Proof. For any p such that $1 \leqslant p < +\infty$, we have

$$\left\|\sqrt{b(y)} <\cdot> \Delta\left(\frac{1}{\sqrt{b}}\right) g(\cdot - y)\right\|_{L^p} \leqslant \|b\|_{L^\infty}^{\frac{1}{2}} \left\|g(\cdot - y) <\cdot> \Delta\left(\frac{1}{\sqrt{b}}\right)\right\|_{L^p}$$

and

$$\begin{aligned}
&\left\|<\cdot> g(\cdot - y)\Delta\left(\frac{1}{\sqrt{b}}\right)\right\|_{L^p}^p \\
&\leqslant \int_{B(y,1)} <x>^p |g(x - y)|^p \left|\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\right|^p \mathrm{d}x \\
&\quad + \int_{B(y,1)^c} <x>^p |g(x - y)|^p \left|\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\right|^p \mathrm{d}x \\
&\leqslant C\|g\|_{L^p(B(0,1))}^p \left\|<\cdot> \Delta\left(\frac{1}{\sqrt{b}}\right)\right\|_{L^\infty}^p \\
&\quad + \int_{B(y,1)^c} (1 + |x|^2)^{\frac{p}{2}} (|x| + |y|)^p \left|\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\right|^p \mathrm{d}x.
\end{aligned}$$

By Assumption 3.1.3, we have that

$$\int_{B(y,1)^c} (1 + |x|^2)^{\frac{p}{2}} (|x| + |y|)^p \left|\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\right|^p \mathrm{d}x$$

$$\leqslant \int_{\mathbb{R}^2} \frac{(1+|x|^2)^{\frac{p}{2}}(|x|+|y|)^p}{(1+|x|)^{(4+\gamma)p}} \, \mathrm{d}x$$
$$\leqslant C_b(1+|y|)^p.$$

Therefore we can apply Proposition 3.2.2 to show that there exists a solution $S_b(\cdot, y)$ of (3.2.9) in $W^{2,p}_{-1}(\mathbb{R}^2)$, unique up to a constant. Since $<x>\geqslant 1$ we also have that

$$\left\| \sqrt{b(y)} g(\cdot - y) \Delta \left(\frac{1}{\sqrt{b}} \right) \right\|_{L^p} \leqslant C_{b,p}(1+|y|).$$

By Lemma 3.2.3, there exists p_0 such that for any $2 < p \leqslant p_0$ and $0 < s < 1$:

$$\begin{aligned} \|\nabla_x S_b(\cdot, y)\|_{L^p} &\leqslant \left\| \sqrt{b(y)} \Delta \left(\frac{1}{\sqrt{b}} \right) g(\cdot - y) \right\|_{L^{\frac{2p}{p+2}}} \\ &\leqslant C_{b,p}(1+|y|) \end{aligned}$$

and

$$\begin{aligned} |\nabla_x S_b(\cdot, y)|_{\mathcal{C}^{0,s}} &\leqslant C_s \left\| \sqrt{b(y)} \Delta \left(\frac{1}{\sqrt{b}} \right) g(\cdot - y) \right\|_{L^{\frac{2}{1-s}}} \\ &\leqslant C_{b,s}(1+|y|) \end{aligned}$$

that is the second inequality of Claim (2). Using that

$$\|\cdot\|_{L^\infty} \leqslant C(\|\cdot\|_{L^p} + |\cdot|_{\mathcal{C}^{0,s}})$$

(see for example the proof of Morrey's embedding theorem in [14, Theorem 9.12]), we get the bound we want on $\nabla_x S_b$:

$$\|\nabla_x S_b(\cdot, y)\|_{L^\infty} \leqslant C_b(1+|y|).$$

If we interpolate the inequalities on $\|\nabla_x S_b(\cdot, y)\|_{L^\infty}$ and $\|\nabla_x S_b(\cdot, y)\|_{L^p}$ for $2 < p \leqslant p_0$ we find that for any $p > 2$,

$$\|\nabla_x S_b(\cdot, y)\|_{L^p} \leqslant C_{b,p}(1+|y|).$$

For the first inequality of Claim (2), let us consider z such that $|z|$ is small and remark that $S_b(x, y+z) - S_b(x, y)$ solves

$$\begin{aligned} &\operatorname{div}\left(\frac{1}{b} (\nabla_x S_b(\cdot, y+z) - \nabla_x S_b(\cdot, y)) \right) \\ &\quad = \left(\sqrt{b(y+z)} g(y+z-\cdot) - \sqrt{b(y)} g(y-\cdot) \right) \Delta \left(\frac{1}{\sqrt{b}} \right). \end{aligned}$$

Let us find a bound for the second member in L^p:

$$\left(\sqrt{b(y+z)} g(y+z-x) - \sqrt{b(y)} g(y-x) \right) \Delta \left(\frac{1}{\sqrt{b}} \right)(x)$$

$$
\begin{aligned}
=&(\sqrt{b(y+z)}-\sqrt{b(y)})g(y-x)\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\\
&+\sqrt{b(y+z)}(g(y+z-x)-g(y-x))\Delta\left(\frac{1}{\sqrt{b}}\right)(x).
\end{aligned}
$$

For the first term,

$$
\left|(\sqrt{b(y+z)}-\sqrt{b(y)})g(y-x)\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\right| \leqslant C_b|z|\left|g(y-x)\Delta\left(\frac{1}{\sqrt{b}}\right)\right|
$$

and we can bound its L^p norms by $C_b(1+|y|)|z|$ as in the proof of Claim (1). For the second term,

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left|\sqrt{b(y+z)}(g(y+z-x)-g(y-x))\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\right|^p \mathrm{d}x\\
\leqslant& C_b\int_{\mathbb{R}^2}\left|(g(x+z)-g(x))\Delta\left(\frac{1}{\sqrt{b}}\right)(y-x)\right|^p \mathrm{d}x\\
\leqslant& C_b\int_{B(0,|z|^\alpha)}|g(x+z)-g(x)|^p \,\mathrm{d}x\\
&+C_b\int_{B(0,|z|^\alpha)^c}|g(x+z)-g(x)|^p\left|\Delta\left(\frac{1}{\sqrt{b}}\right)(y-x)\right|^p \mathrm{d}x
\end{aligned}
$$

for any $0<\alpha<1$. Now, if $|z|$ is small enough,

$$
\int_{B(0,|z|^\alpha)}|g(x+z)-g(x)|^p \,\mathrm{d}x \leqslant C\int_{B(0,|z|^\alpha)} g(x+z)^p+g(x)^p \,\mathrm{d}x.
$$

Now we use a classical rearrangement procedure to bound

$$
\begin{aligned}
&\int_{B(0,|z|^\alpha)} g(x+z)^p-\int_{B(0,|z|^\alpha)} g(x)^p \,\mathrm{d}x\\
&\qquad=\int_{B(z,|z|^\alpha)} g(x)^p-\int_{B(0,|z|^\alpha)} g(x)^p \,\mathrm{d}x\\
&\qquad=\int_{B(0,|z|^\alpha)} g(x)^p(\mathbf{1}_{B(z,|z|^\alpha)}(x)-1)\,\mathrm{d}x\\
&\qquad\quad+\int_{B(0,|z|^\alpha)^c\cap B(z,|z|^\alpha)} g(x)^p \,\mathrm{d}x
\end{aligned}
$$

Now remark that for $x\in B(0,|z|^\alpha)$, $g(x)^p \geqslant -\frac{1}{2\pi}\ln^p(|z|^\alpha)$ and therefore

$$
\begin{aligned}
&\int_{B(0,|z|^\alpha)} g(x)^p(\mathbf{1}_{B(z,|z|^\alpha)}(x)-1)\,\mathrm{d}x\\
&\qquad\leqslant -\frac{1}{2\pi}\ln^p(|z|^\alpha)\int_{B(0,|z|^\alpha)}(\mathbf{1}_{B(z,|z|^\alpha)}(x)-1)\,\mathrm{d}x
\end{aligned}
$$

$$\leqslant -\frac{1}{2\pi}\ln^p(|z|^\alpha)(|B(0,|z|^\alpha)\cap B(z,|z|^\alpha)| - |B(0,|z|^\alpha)|)$$

and on $B(0,|z|^\alpha)^c$, $g(x) \leqslant -\frac{1}{2\pi}\ln(|z|^\alpha)$ so

$$\int_{B(0,|z|^\alpha)^c\cap B(z,|z|^\alpha)} g(x)^p \,\mathrm{d}x \leqslant -\frac{1}{2\pi}\ln^p(|z|^\alpha)|B(0,|z|^\alpha)^c \cap B(z,|z|^\alpha)|.$$

We get

$$\int_{B(0,|z|^\alpha)} g(x+z)^p - \int_{B(0,|z|^\alpha)} g(x)^p \,\mathrm{d}x \leqslant 0$$

and therefore

$$\begin{aligned}\int_{B(0,|z|^\alpha)} |g(x+z)-g(x)|^p \,\mathrm{d}x &\leqslant 2\int_{B(0,|z|^\alpha)} g(x)^p \,\mathrm{d}x \\ &\leqslant C|z|^{2\alpha}\int_{B(0,1)} g(|z|^\alpha y)^p \,\mathrm{d}y \\ &\leqslant C|z|^{2\alpha}\int_{B(0,1)} (\alpha g(z)+g(y))^p \,\mathrm{d}y \\ &\leqslant C_b|z|^{2\alpha} g(z)^p.\end{aligned}$$

Now if $|z|$ is small enough,

$$\begin{aligned}&C_b\int_{B(0,|z|^\alpha)^c} |g(x+z)-g(x)|^p \,\mathrm{d}x \left|\Delta\left(\frac{1}{\sqrt{b}}\right)(y-x)\right|^p \\ &\leqslant C_b\left(|z|\frac{C}{|z|^\alpha}\right)^p \int_{\mathbb{R}^2}\left|\Delta\left(\frac{1}{\sqrt{b}}\right)(y-x)\right|^p \,\mathrm{d}x \\ &\leqslant C_b|z|^{p(1-\alpha)}\end{aligned}$$

by Assumption 3.1.3. Finally, using Lemma 3.2.3 as for the first claim, we get that for any $0<\alpha<1$ and some $p>2$,

$$|\nabla_x S_b(x,y+z) - \nabla_x S_b(x,y)| \leqslant C_b(1+|y|)|z| + C_b(|z|^{\frac{2\alpha}{p}} g(z) + |z|^{1-\alpha}).$$

Dividing both sides by $|z|^s$ for s small enough proves the first inequality of Claim (2). □

With this lemma we are now able to construct the lake kernel. The construction is similar to the one established in [22, Proposition 3.1] for bounded domains.

Proposition 3.2.8. *There exists a symmetric solution S_b of Equation (3.2.9) such that $S_b(0,0)=0$. We define g_b as*

$$g_b(x,y) := \sqrt{b(x)b(y)}\,g(x-y) + S_b(x,y). \tag{3.2.10}$$

Let $\omega \in L^\infty$ with compact support. We define

$$G_b[\omega](x) = \int_{\mathbb{R}^2} g_b(x,y)\,\mathrm{d}\omega(y). \tag{3.2.11}$$

Then $G_b[\omega]$ is a distributional solution of (3.2.2).

Moreover for $2 < p < +\infty$, $G_b[\omega]$ is the unique solution (up to a constant) of (3.2.2) in $W^{2,p}_{-1}(\mathbb{R}^2)$ given by Proposition 3.2.2.

Proof of Proposition 3.2.8. Let us first define

$$g_b(x,y) := \sqrt{b(x)b(y)}g(x-y) + S_b(x,y)$$

where S_b is a solution of Equation (3.2.9) given by Proposition 3.2.2 (not necessarily symmetric). Then we have the following result:

Claim 3.2.9. If φ is smooth with compact support, then

$$-\int_{\mathbb{R}^2} g_b(x,y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x = \varphi(y).$$

Proof of Claim 3.2.9. We have

$$\begin{aligned}
-\int_{\mathbb{R}^2} & g_b(x,y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^2} \sqrt{b(x)b(y)}g(x-y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x \\
&\quad -\int_{\mathbb{R}^2} S_b(x,y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x \\
&=: T_1 + T_2.
\end{aligned}$$

We have

$$\begin{aligned}
T_1 =& -\sqrt{b(y)}\int_{\mathbb{R}^2} \sqrt{b(x)}g(x-y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x \\
=& \sqrt{b(y)}\int_{\mathbb{R}^2} g(x-y)\frac{1}{2b(x)\sqrt{b(x)}}\nabla b(x)\cdot\nabla\varphi(x)\,\mathrm{d}x \\
&+ \sqrt{b(y)}\int_{\mathbb{R}^2} \frac{1}{\sqrt{b(x)}}\nabla g(x-y)\cdot\nabla\varphi(x)\,\mathrm{d}x \\
=&: L_1 + L_2.
\end{aligned}$$

Integrating by parts in the first integral we get

$$\begin{aligned}
L_1 =& -\sqrt{b(y)}\int_{\mathbb{R}^2} \varphi(x)\frac{1}{2b(x)\sqrt{b(x)}}\nabla g(x-y)\cdot\nabla b(x)\,\mathrm{d}x \\
&- \sqrt{b(y)}\int_{\mathbb{R}^2} \varphi(x)g(x-y)\,\mathrm{div}\left(\frac{1}{2b\sqrt{b}}\nabla b\right)(x)\,\mathrm{d}x.
\end{aligned}$$

For L_2, we use

$$\nabla\left(\frac{1}{\sqrt{b}}\varphi\right) = \frac{1}{\sqrt{b}}\nabla\varphi - \varphi\frac{1}{2b\sqrt{b}}\nabla b$$

to get

$$\begin{aligned}
L_2 =& \sqrt{b(y)}\int_{\mathbb{R}^2}\varphi(x)\frac{1}{2b(x)\sqrt{b(x)}}\nabla b(x)\cdot\nabla g(x-y)\,\mathrm{d}x \\
&+ \sqrt{b(y)}\int_{\mathbb{R}^2}\nabla\left(\frac{1}{\sqrt{b(x)}}\varphi(x)\right)\cdot\nabla g(x-y)\,\mathrm{d}x \\
=& \sqrt{b(y)}\int_{\mathbb{R}^2}\varphi(x)\frac{1}{2b(x)\sqrt{b(x)}}\nabla b(x)\cdot\nabla g(x-y)\,\mathrm{d}x \\
&+ \varphi(y)
\end{aligned}$$

since $-\Delta_x g(x-y) = \delta_y$ distributionally. Now let us compute T_2:

$$\begin{aligned}
T_2 &= -\int_{\mathbb{R}^2} S_b(x,y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^2}\mathrm{div}\left(\frac{1}{b}\nabla_x S_b(\cdot,y)\right)(x)\varphi(x)\,\mathrm{d}x \\
&= -\sqrt{b(y)}\int_{\mathbb{R}^2} g(x-y)\Delta\left(\frac{1}{\sqrt{b}}\right)(x)\varphi(x)\,\mathrm{d}x
\end{aligned}$$

where we used that S_b is a solution of (3.2.9) in the last line. Now just remark that

$$\Delta\left(\frac{1}{\sqrt{b}}\right) = -\,\mathrm{div}\left(\frac{1}{2b\sqrt{b}}\nabla b\right)$$

and thus adding L_1 and L_2 we get

$$-\int_{\mathbb{R}^2} g_b(x,y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x = \varphi(y)$$

and we get the proof of Claim 3.2.9. □

Now let $\omega \in L^\infty(\mathbb{R}^2)$ with compact support. We have

$$\begin{aligned}
-\int_{\mathbb{R}^2}&\left(\int_{\mathbb{R}^2} g_b(x,y)\omega(y)\,\mathrm{d}y\right)\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} g_b(x,y)\,\mathrm{div}\left(\frac{1}{b}\nabla\varphi\right)(x)\,\mathrm{d}x\right)\omega(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^2}\varphi(y)\omega(y)\,\mathrm{d}y
\end{aligned}$$

where we used Claim 3.2.9 in the last equality. Therefore $G_b[\omega]$ is a distributional solution of (3.2.2).

Now we prove that with this kernel we recover solutions in the sense of Proposition 3.2.2:

Claim 3.2.10. Let $\omega \in L^\infty$ with compact support, then for all $p \in (2, +\infty)$, we have that $\nabla G_b[\omega] \in L^p$. Moreover if ψ is the solution of (3.2.2) given by Proposition (3.2.2), then $\psi = G_b[\omega]$ up to a constant.

Proof of the Claim 3.2.10. We have:

$$\begin{aligned}\nabla G_b[\omega](x) = &\int_{\mathbb{R}^2} \frac{\nabla b(x)}{2\sqrt{b(x)}} \sqrt{b(y)} g(x-y)\omega(y)\,\mathrm{d}y \\ &+ \int_{\mathbb{R}^2} \sqrt{b(x)b(y)} \nabla g(x-y)\omega(y)\,\mathrm{d}y \\ &+ \int_{\mathbb{R}^2} \nabla_x S_b(x,y)\omega(y)\,\mathrm{d}y \\ =: &T_1 + T_2 + T_3.\end{aligned}$$

Now,

$$\begin{aligned}|T_1| \leqslant &C_b |\nabla b(x)| \bigg(\int_{B(x,1)} |(\ln|x-y|)\omega(y)|\,\mathrm{d}y \\ &+ \int_{\mathrm{supp}(\omega)\setminus B(x,1)} (|x|+|y|)|\omega(y)|\,\mathrm{d}y \bigg) \\ \leqslant &C_b \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} (1+|x|)^{-(3+\gamma)}\end{aligned}$$

by Assumption 3.1.3. Hence $T_1 \in L^p$. For the second term, we have

$$T_2 = \sqrt{b(x)} \nabla g * (\sqrt{b}\omega)$$

and therefore $T_2 \in L^p$ by Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]). For the third term,

$$|T_3| \leqslant \left(\int_{\mathbb{R}^2} |\omega| \right) \int_{\mathbb{R}^2} |\nabla_x S_b(x,y)| \frac{|\omega(y)|\,\mathrm{d}y}{\int |\omega|}$$

and thus by Jensen inequality

$$\|T_3\|_{L^p}^p \leqslant \left(\int_{\mathbb{R}^2} |\omega| \right)^{p-1} \iint_{\mathbb{R}^2\times\mathbb{R}^2} |\nabla_x S_b(x,y)|^p |\omega(y)|\,\mathrm{d}y\,\mathrm{d}x.$$

We have that

$$\|\nabla S_b(\cdot, y)\|_{L^p} \leqslant C_b(1+|y|)$$

by Claim (1) of Lemma 3.2.7. Therefore

$$\|T_3\|_{L^p}^p \leqslant C_b \left(\int_{\mathbb{R}^2} |\omega| \right)^{p-1} \int_{\mathbb{R}^2} (1+|y|)^p |\omega(y)|\,\mathrm{d}y$$

and it follows that $\nabla G_b[\omega] \in L^p$. By Proposition 3.2.2 we get that $G_b[\omega] = \psi$ up to a constant. □

We are only left to justify that there exists a symmetric solution of (3.2.9). Consider ω_1, ω_2 two smooth functions with average zero, then by Claim 3.2.10, we have

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_1(x)\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^2} (\psi_2(x)+C)\omega_1(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^2} \psi_2(x)\,\mathrm{div}\left(\frac{1}{b}\nabla\psi_1\right)(x)\,\mathrm{d}x\end{aligned}$$

where ψ_i is the solution of

$$-\,\mathrm{div}\left(\frac{1}{b}\nabla\psi_i\right) = \omega_i$$

given by Proposition 3.2.2. If $R>0$, we have that

$$\begin{aligned}-\int_{B(0,R)} \psi_2(x)\,\mathrm{div}\left(\frac{1}{b}\nabla\psi_1\right)(x)\,\mathrm{d}x = &-\int_{\partial B(0,R)} \frac{1}{b}\psi_2\nabla\psi_1\cdot\mathrm{d}\vec{S} \\ &+\int_{B(0,R)} \frac{1}{b}\nabla\psi_2\cdot\nabla\psi_1.\end{aligned}$$

Using Proposition 3.2.5, we obtain

$$\left|\int_{\partial B(0,R)} \frac{1}{b}\psi_2\nabla\psi_1\cdot\mathrm{d}\vec{S}\right| \leqslant 2\pi R\left\|b^{-1}\right\|_{L^\infty} C(1+R^\delta)\frac{C}{R^2} \underset{R\to+\infty}{\longrightarrow} 0$$

and therefore

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_1(x)\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2}\frac{1}{b}\nabla\psi_2\cdot\nabla\psi_1$$

which is a symmetric expression of ψ_1 and ψ_2. It follows that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega_1(x)\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(y,x)\omega_1(x)\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since $\sqrt{b(x)b(y)}g(x-y)$ is symmetric we get that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} S_b(x,y)\omega_1(x)\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^2\times\mathbb{R}^2} S_b(y,x)\omega_1(x)\omega_2(y)\,\mathrm{d}x\,\mathrm{d}y$$

for any ω_1, ω_2 smooth with compact suport and average zero. Let us define

$$A(x,y) := S_b(x,y) - S_b(y,x).$$

Now we fix χ, ω_1, ω_2 smooth functions with compact support such that $\int_{\mathbb{R}^2}\omega_2 = 0$ and $\int_{\mathbb{R}^2}\chi = 1$. Remark that we no longer assume that $\int_{\mathbb{R}^2}\omega_1 = 0$. We define

$$A_2(x) := \int_{\mathbb{R}^2} A(x,y)\omega_2(y)\,\mathrm{d}y.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^2} A_2\omega_1 &= \int_{\mathbb{R}^2} A_2\left(\omega_1 - \left(\int_{\mathbb{R}^2}\omega_1\right)\chi\right) + \left(\int_{\mathbb{R}^2}\omega_1\right)\int_{\mathbb{R}^2} A_2\chi \\ &= 0 + \left(\int_{\mathbb{R}^2}\omega_1\right)\int_{\mathbb{R}^2} A_2\chi.\end{aligned}$$

Thus A_2 is constant so for every $x \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \nabla_x A(x,y)\omega_2(y)\,\mathrm{d}y = 0$$

for all ω_2 with mean zero and therefore $\nabla_x A(x,y) = U(x)$. It follows that $A(x,y) = c(x) + d(y)$. Since $A(x,y) = -A(y,x)$, we have $d = -c$. Now let us set $\widetilde{S}_b(x,y) := S_b(x,y) + c(y)$. We have:

$$\begin{aligned}\widetilde{S}_b(x,y) - \widetilde{S}_b(y,x) &= S_b(x,y) - S_b(y,x) + c(y) - c(x) \\ &= c(x) - c(y) + c(y) - c(x) \\ &= 0\end{aligned}$$

which proves that $\widetilde{S}_b$ a symmetric solution of (3.2.9). Up to adding a constant we can also assume that $\widetilde{S}_b(0,0) = 0$. □

The symmetry of S_b allows us to obtain more regularity estimates:

Lemma 3.2.11. *Let S_b be the symmetric solution of Equation (3.2.9) given by Proposition 3.2.8, then*

1. *S_b is smooth on $\mathbb{R}^2 \times \mathbb{R}^2 \backslash \{(x,x)\ ;\ x \in \mathbb{R}^2\}$.*
2. *$|S_b(x,y)| \leqslant C_b(1 + |x|^2 + |y|^2)$.*

Proof. For $0 < r < R$, we define $C(y,r,R) := B(y,R)\backslash B(y,r)$. We have that $S_b(\cdot,y)$ is a solution of

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{b}\nabla S_b(\cdot,y)\right) = g(\cdot - y)\sqrt{b(y)}\Delta\left(\dfrac{1}{\sqrt{b}}\right) & \text{in } C(y,r,R) \\ S_b(\cdot,y) = S_b(\cdot,y) \in \mathcal{C}^{0,s} \qquad \text{in } \partial C(y,r,R). \end{cases}$$

Thus by elliptic regularity (see for example [32, Theorem 6.13]) we obtain that $S_b(\cdot,y) \in \mathcal{C}^{2,s}(\mathring{C}(y,r,R))$ for all $y \in \mathbb{R}^2$ and $0 < r < R$. By symmetry we get that S_b is $\mathcal{C}^{2,s}$ on

$$\mathbb{R}^2 \times \mathbb{R}^2 \backslash \{(x,x)\ ;\ x \in \mathbb{R}^2\}.$$

We can iterate the argument by writing the elliptic system satisfied by the derivatives of S_b to show that S_b is smooth on

$$\mathbb{R}^2 \times \mathbb{R}^2 \backslash \{(x,x)\ ;\ x \in \mathbb{R}^2\}.$$

The second claim is just a consequence of Lemma 3.2.7, since

$$\begin{aligned}|S_b(x,y)| &\leqslant |S_b(0,0) - S_b(x,0)| + |S_b(x,0) - S_b(x,y)| \\ &\leqslant |S_b(0,0) - S_b(x,0)| + |S_b(0,x) - S_b(y,x)| \\ &\leqslant \|\nabla_x S_b(\cdot,0)\|_{L^\infty}\, |x| + \|\nabla_x S_b(\cdot,x)\|_{L^\infty}\, |y| \\ &\leqslant C_b|x| + C_b(1+|x|)|y| \\ &\leqslant C_b(1+|x|^2+|y|^2).\end{aligned}$$

□

We finish this subsection by giving a straightforward consequence of Proposition 3.2.5 and [27, Lemma 2.7] which will be useful to deal with the regularisation of the dirac mass we will introduce in Subsection 3.2.4 and use in Sections 3.5 and 3.6.

Lemma 3.2.12. *$\mu \mapsto \nabla G_b[\mu]$ extends into a bounded operator from $\dot{H}^{-1}$ to L^2.*

Proof. Let μ be a smooth function with compact support and average zero. By Proposition 3.2.5, $\nabla G_b[\mu] \in L^2$ and therefore it follows by [27, Lemma 2.7] that

$$\|\nabla G_b[\mu]\|_{L^2} \leqslant C_b \|\mu\|_{\dot{H}^{-1}}$$

and the lemma follows from the density of smooth functions with compact support and average zero in $\dot{H}^{-1}$. □

3.2.4 Regularisations of the Coulomb kernel and the dirac mass

To study our modulated energy we will need to have suitable regularisations of g and of the dirac mass δ_y. For that purpose, let us first define $g^{(\eta)}$ for any $0 < \eta < 1$ as

$$g^{(\eta)}(x) := \begin{cases} -\dfrac{1}{2\pi}\ln(\eta) & \text{if } |x| \leqslant \eta \\ g(x) & \text{if } |x| \geqslant \eta \end{cases} \tag{3.2.12}$$

and we define $\delta_y^{(\eta)}$ as the uniform probability measure on the circle $\partial B(y,\eta)$. We recall that by Lemma 2.3.1, for any $0 < \eta < 1$ and $y \in \mathbb{R}^2$,

$$\int g(x-z)\,\mathrm{d}\delta_y^{(\eta)}(z) = g^{(\eta)}(x-y).$$

We also define

$$\widetilde{\delta}_y^{(\eta)} := m_b(y,\eta)\frac{\mathrm{d}\delta_y^{(\eta)}}{\sqrt{b}} \tag{3.2.13}$$

where

$$m_b(y,\eta) := \left(\int \frac{\mathrm{d}\delta_y^{(\eta)}}{\sqrt{b}} \right)^{-1}. \tag{3.2.14}$$

Remark that

$$m_b(y,\eta) - \sqrt{b(y)} = m_b(y,\eta)\sqrt{b(y)} \left(\frac{1}{\sqrt{b(y)}} - \int \frac{\mathrm{d}\delta_y^{(\eta)}(z)}{\sqrt{b(z)}} \right)$$

and therefore by Assumption 3.1.3

$$|m_b(y,\eta) - \sqrt{b(y)}| \leqslant C_b \eta. \tag{3.2.15}$$

3.3 Point vortices

To prove Theorem 3.1.6 we will need to control the evolution of the interaction energy and of the moment of inertia. We recall that the moment of inertia is not conserved for the lake equations, nor for the point vortex system. Due to the self-interactions, the interaction energy E_N is also not conserved.

The following proposition gives bounds on the interaction energy and on the moment of inertia and the global well-posedness of the lake point-vortex system (3.1.1).

Proposition 3.3.1. *Let $T > 0$ and $(q_1^0, ..., q_N^0)$ be such that $q_i^0 \neq q_j^0$ if $i \neq j$. There exists a unique smooth solution of (3.1.1) on $[0,T]$. Moreover, we have the following estimates:*

$$|E_N(t)| \leqslant e^{C_b(1+\alpha_N)t}(|E_N(0)| + I_N(0) + 1) \tag{3.3.1}$$

$$I_N(t) \leqslant e^{C_b(1+\alpha_N)t}(|E_N(0)| + I_N(0) + 1). \tag{3.3.2}$$

We also have similar estimates for the rescaled moment of inertia and for the interaction energy:

$$|\overline{E_N}(t)| \leqslant e^{C_b(1+\alpha_N^{-1})t}(|\overline{E_N}(0)| + \overline{I_N}(0) + 1) \tag{3.3.3}$$

$$\overline{I_N}(t) \leqslant e^{C_b(1+\alpha_N^{-1})t}(|\overline{E_N}(0)| + \overline{I_N}(0) + 1). \tag{3.3.4}$$

Proof. Since b is regular (see Assumption 3.1.3) and $S_b, g, \nabla g$ are regular outside of the diagonal (see Claim (1) of Lemma 3.2.11), System (3.1.1) is well-posed up to the first collision time by Cauchy-Lipschitz theorem. We will first prove the bounds on E_N and I_N and then deduce that there is no collision between the points (this is the classical strategy to prove that the Euler point vortex system is well-posed when all the vorticities are positive, as explained for example in [63, Chapter 4.2]). Let us assume that there is no collision up to some time $T^* \leqslant T$.

We first compute the time derivative of E_N. Since g_b is symmetric, we have

$$\begin{aligned}
\dot{E}_N =& \frac{1}{N^2} \sum_{i=1}^N \Big(\sum_{\substack{j=1\\ j\neq i}}^N \dot{q}_i \cdot \nabla_x g_b(q_i,q_j) + \dot{q}_j \nabla_y g_b(q_i,q_j) \Big) \\
=& \frac{2}{N^2} \sum_{i=1}^N \Big(-\alpha_N \frac{\nabla^\perp b(q_i)}{b(q_i)} - \frac{1}{N b(q_i)} \sum_{\substack{k=1\\ k\neq i}}^N \nabla_x^\perp g_b(q_i,q_k) \Big) \cdot \sum_{\substack{j=1\\ j\neq i}}^N \nabla_x g_b(q_i,q_j) \\
=& -\frac{2\alpha_N}{N^2} \sum_{i=1}^N \frac{\nabla^\perp b(q_i)}{b(q_i)} \cdot \Big(\frac{\sqrt{b(q_i)}}{2\sqrt{b(q_j)}} g(q_i-q_j) \nabla b(q_j) \\
&+ \sqrt{b(q_i)b(q_j)} \nabla g(q_i-q_j) + \nabla_x S_b(q_i,q_j) \Big)
\end{aligned}$$

and thus we get that

(3.3.5)
$$\dot{E}_N = -\frac{2\alpha_N}{N^2} \sum_{i=1}^N \frac{\nabla^\perp b(q_i)}{b(q_i)} \cdot \Big(\sum_{\substack{j=1\\ j\neq i}}^N \sqrt{b(q_i)b(q_j)} \nabla g(q_i-q_j) + \nabla_x S_b(q_i,q_j) \Big).$$

Now let us bound the right-handside of the upper equality. Using Claim (1) of Lemma 3.2.7 and Assumption 3.1.3, we have

$$\left| \frac{\nabla^\perp b(q_i)}{b(q_i)} \cdot \nabla_x S_b(q_i,q_j) \right| \leqslant C_b(1+|q_j|)$$

and thus

$$\left| \frac{2\alpha_N}{N^2} \sum_{i=1}^N \frac{\nabla^\perp b(q_i)}{b(q_i)} \cdot \sum_{\substack{j=1\\ j\neq i}}^N \nabla_x S_b(q_i,q_j) \right| \leqslant C_b \alpha_N (1+I_N). \tag{3.3.6}$$

Now remark that

$$\begin{aligned}
&\frac{2\alpha_N}{N^2} \sum_{i=1}^N \frac{\nabla^\perp b(q_i)}{b(q_i)} \cdot \Big(\sum_{\substack{j=1\\ j\neq i}}^N \sqrt{b(q_i)b(q_j)} \nabla g(q_i-q_j) \Big) \\
&\quad = \frac{\alpha_N}{N^2} \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \left(\sqrt{\frac{b(q_j)}{b(q_i)}} \nabla^\perp b(q_i) - \sqrt{\frac{b(q_i)}{b(q_j)}} \nabla^\perp b(q_j) \right) \cdot \nabla g(q_i-q_j).
\end{aligned}$$

Moreover,

$$\sqrt{\frac{b(q_j)}{b(q_i)}}\nabla^\perp b(q_i)-\sqrt{\frac{b(q_i)}{b(q_j)}}\nabla^\perp b(q_j)=\sqrt{\frac{b(q_j)}{b(q_i)}}(\nabla^\perp b(q_i)-\nabla^\perp b(q_j)) + \frac{b(q_j)-b(q_i)}{\sqrt{b(q_i)b(q_j)}}\nabla^\perp b(q_j)$$

and thus using the Lipschitz regularity of b and ∇b (see Assumption 3.1.3) and $|\nabla g(q_i-q_j)|=C|q_i-q_j|^{-1}$ we get that

$$\left|\frac{2\alpha_N}{N^2}\sum_{i=1}^N\frac{\nabla^\perp b(q_i)}{b(q_i)}\cdot\left[\sum_{\substack{j=1\\j\neq i}}^N\sqrt{b(q_i)b(q_j)}\nabla g(q_i-q_j)\right]\right|\leqslant C_b\alpha_N. \tag{3.3.7}$$

Combining inequalities (3.3.6) and (3.3.7) we get that

$$|\dot{E}_N|\leqslant C_b(1+I_N)\alpha_N. \tag{3.3.8}$$

Now we compute the time derivative of I_N:

$$\begin{aligned}
\dot{I}_N=&\frac{2}{N}\sum_{i=1}^N q_i\cdot\dot{q}_i\\
=&-\frac{2\alpha_N}{N}\sum_{i=1}^N q_i\cdot\frac{\nabla^\perp b(q_i)}{b(q_i)}\\
&-\frac{2}{N}\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N\frac{\sqrt{b(q_j)}}{2b(q_i)\sqrt{b(q_i)}}g(q_i-q_j)q_i\cdot\nabla^\perp b(q_i)\\
&-\frac{2}{N}\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N\frac{\sqrt{b(q_i)b(q_j)}}{b(q_i)}q_i\cdot\nabla^\perp g(q_i-q_j)\\
&-\frac{2}{N}\sum_{i=1}^N\sum_{\substack{j=1\\j\neq i}}^N q_i\cdot\nabla_x^\perp S_b(q_i,q_j)\\
=:&2(T_1+T_2+T_3+T_4).
\end{aligned}$$

Using Assumption 3.1.3 we have

$$|T_1|\leqslant C_b\alpha_N. \tag{3.3.9}$$

For the second term, using Assumption 3.1.3 we have

$$|T_2|\leqslant\frac{C_b}{N^2}\sum_{i=1}^N\left(\sum_{\substack{j=1\\j\neq i}}^N|g(q_i-q_j)|\right)$$

$$\leqslant \frac{C_b}{N^2}\sum_{i=1}^N\Bigg(\sum_{\substack{j=1\\ j\neq i}}^N g(q_i-q_j)\mathbf{1}_{|q_i-q_j|\leqslant 1}+|q_i|^2+|q_j|^2\Bigg)$$

$$\leqslant C_bI_N+\frac{C_b}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |q_i-q_j|\leqslant 1}} g(q_i-q_j).$$

Now by Assumption 3.1.3, we have that

$$\begin{aligned}\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |q_i-q_j|\leqslant 1}} g(q_i-q_j)\leqslant & \frac{C_b}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\Bigg(\sqrt{b(q_i)b(q_j)}g(q_i-q_j)\\ &+S_b(q_i,q_j)\Bigg)+\frac{C_b}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |q_i-q_j|\geqslant 1}}|g(q_i-q_j)|\\ &+\frac{C_b}{N^2}\sum_{1\leqslant i\neq j\leqslant N}|S_b(q_i,q_j)|\\ \leqslant & C_b\Bigg(E_N+\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |q_i-q_j|\geqslant 1}}|g(q_i-q_j)|\\ &+\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}|S_b(q_i,q_j)|\Bigg).\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{C_b}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |q_i-q_j|\geqslant 1}}|g(q_i-q_j)|&\leqslant \frac{C_b}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\ |q_i-q_j|\geqslant 1}}|q_i|^2+|q_j|^2\\ &\leqslant C_bI_N\end{aligned}$$

and using Claim (2) of Lemma 3.2.11,

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}|S_b(q_i,q_j)|\leqslant \frac{C_b}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(1+|q_i|^2+|q_j|^2)\leqslant C_b(1+I_N).$$

Therefore

$$|T_2|\leqslant C_b(1+|E_N|+I_N). \tag{3.3.10}$$

For the third term we write

$$T_3=-\frac{1}{N^2}\sum_{i=1}^N\sum_{\substack{j=1\\ j\neq i}}^N\frac{\sqrt{b(q_j)}-\sqrt{b(q_i)}}{\sqrt{b(q_i)}}\nabla^\perp g(q_i-q_j)\cdot q_i$$

$$- \frac{1}{2N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla^\perp g(q_i - q_j) \cdot (q_i - q_j)$$

$$= - \frac{1}{N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\sqrt{b(q_j)} - \sqrt{b(q_i)}}{\sqrt{b(q_i)}} \nabla^\perp g(q_i - q_j) \cdot q_i - 0$$

and thus using the Lipschitz regularity of b (see Assumption 3.1.3) we get

$$|T_3| \leqslant C_b(1 + I_N). \tag{3.3.11}$$

For the fourth term, using Claim (1) of Lemma 3.2.7 we get

$$\begin{aligned} |T_4| &= \left| - \frac{1}{N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{b(q_i)} q_i \cdot \nabla_x^\perp S_b(q_i, q_j) \right| \\ &\leqslant C_b \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |q_i|(1 + |q_j|) \\ &\leqslant C_b(1 + I_N). \end{aligned} \tag{3.3.12}$$

Combining with inequalities (3.3.9), (3.3.10), (3.3.11) and (3.3.12) we get that

$$|\dot{I}_N| \leqslant C_b(1 + \alpha_N + |I_N| + |E_N|). \tag{3.3.13}$$

Let us write $U_N := (E_N, I_N)$. By equations (3.3.8) and (3.3.13) we have

$$|\dot{U}_N| \leqslant C_b(1 + \alpha_N)(1 + |U_N|)$$

therefore by Grönwall's lemma we have

$$|U_N(t)| \leqslant e^{C_b(1+\alpha_N)t}(|U_N(0)| + 1) - 1$$

from which (3.3.1) and (3.3.2) follows.

Let us use these bounds to prove that there is no collision (and it will follow that System (3.1.1) is globally well-posed). If $i \neq j$, then

$$\begin{aligned} g(|q_i - q_j|) \leqslant & C_b \Bigg(E_N + \frac{1}{N^2} \sum_{1 \leqslant k \neq l \leqslant N} \sum |S_b(q_k, q_l)| \\ & - \frac{1}{N^2} \sum_{\substack{1 \leqslant k \neq l \leqslant N \\ (k,l) \neq (i,j)}} g(q_k - q_l) \Bigg) \end{aligned}$$

$$\leqslant C_b\Big(E_N + \frac{1}{N^2}\sum_{1\leqslant k\neq l\leqslant N}(1+|q_k|^2+|q_l|^2)\Big).$$

where we used Claim (2) of Lemma 3.2.11 and $\ln|x-y| \leqslant |x|+|y|$. Thus by inequalities (3.3.1) and (3.3.2) we get

$$g(|q_i-q_j|) \leqslant C_b(e^{C_b(1+\alpha_N)t}(|E_N(0)|+I_N(0)+1)+1)$$

and therefore

$$|q_i(t)-q_j(t)| \geqslant \exp\Big(-2\pi C_b(e^{C_b(1+\alpha_N)t}(|E_N(0)|+I_N(0)+1)+1)\Big) > 0.$$

It follows that there is no collision on $[0,T]$. The bounds on $\overline{E_N}$ and $\overline{I_N}$ follow directly from Inequalities (3.3.1) and (3.3.2) applied to $t=\alpha_N^{-1}\tau$. □

3.4 Time derivatives of the modulated energies

The time derivatives of $\mathcal{F}_{b,N}$ and of $\overline{\mathcal{F}}_{b,N}$, defined in (3.1.9) and (3.1.10), are given by the two following propositions:

Proposition 3.4.1. *Let ω be a weak solution of (3.1.2) in the sense of Definition 3.1.1, $(q_1,...,q_N)$ be solutions of (3.1.1). We denote*

$$\omega_N = \frac{1}{N}\sum_{i=1}^N \delta_{q_i(t)}.$$

Assume that ω satisfies Assumption 3.1.4. Then $\mathcal{F}_{b,N}$ is Lipschitz and for almost every $t\in[0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{b,N}(t) =$$
$$2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\Big(u(t,x)-\alpha\frac{\nabla^\perp b(x)}{b(x)}\Big)\cdot\nabla_x g_b(x,y)\,\mathrm{d}(\omega(t)-\omega_N(t))^{\otimes 2}(x,y)$$
$$+2(\alpha_N-\alpha)\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\frac{\nabla^\perp b(x)}{b(x)}\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(t,x)\,\mathrm{d}(\omega(t)-\omega_N(t))(y).$$

Proposition 3.4.2. *Let $(\overline{q_1},...,\overline{q_N})$ be solutions of (3.1.4) and $\overline{\omega}$ be a solution of (3.1.5) in the sense of Definition 3.1.2.*

We denote

$$\overline{\omega}_N = \frac{1}{N}\sum_{i=1}^N \delta_{\overline{q_i}(t)}.$$

Assume that $\overline{\omega}$ satisfies Assumption 3.1.4. Denote $v=\nabla G_b[\overline{\omega}]$. Then $\overline{\mathcal{F}}_{b,N}$ is Lipschitz and for almost every $t\in[0,T]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{F}}_{b,N}(t) = -2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x,y)\,\mathrm{d}(\overline{\omega}(t) - \overline{\omega}_N(t))^{\otimes 2}(x,y)$$
$$+ \frac{2}{N^2\alpha_N}\sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \frac{v(t,\overline{q_i})}{b(\overline{q_i})} \cdot \nabla_x g_b(\overline{q_i},\overline{q_j}).$$

Proof of Proposition 3.4.1. We split $\mathcal{F}_{b,N}$ in three terms:

$$\begin{aligned}\mathcal{F}_{b,N} =& \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(t,x)\omega(t,y)\,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{2}{N}\sum_{i=1}^N \int_{\mathbb{R}^2} g_b(x,q_i)\omega(t,x)\,\mathrm{d}x + E_N \\ =&:T_1 + T_2 + E_N.\end{aligned}$$

Let us compute the time derivative of T_1. For that purpose, we will need to regularize the kernel g_b. The regularisation we will use is given by the following Claim:

Claim 3.4.3. There exists a familly of smooth functions $(g_b^\eta)_{0<\eta<1}$ such that:
- $|g_b^\eta(x,y)| \leqslant C_b(|g(x-y)| + 1 + |x|^2 + |y|^2)$
- $|\nabla_x g_b^\eta(x,y)|, |\nabla_y g_b^\eta(x,y)| \leqslant C_b(|x-y|^{-1} + 1 + |x| + |y|)$.
- For any $(x,y) \in (\mathbb{R}^2)^2$ such that $x \neq y$,

$$\begin{aligned} g_b^\eta(x,y) &\underset{\eta\to 0}{\longrightarrow} g_b(x,y) \\ \nabla_x g_b^\eta(x,y) &\underset{\eta\to 0}{\longrightarrow} \nabla_x g_b(x,y) \\ \nabla_y g_b^\eta(x,y) &\underset{\eta\to 0}{\longrightarrow} \nabla_y g_b(x,y). \end{aligned}$$

Proof of Claim 3.4.3. We define

$$g_b^\eta(x,y) = \sqrt{b(x)b(y)}\,g^\eta(x-y) + S_b^\eta(x,y)$$

where g^η is a smooth function satisfying:
- $g^\eta(x) = g(x)$ for $|x| \geqslant \eta$,
- $|g^\eta(x)| \leqslant |g(x)|$,
- $|\nabla g^\eta(x)| \leqslant C|x|^{-1}$.

that we can obtain by extending $\ln|_{x\geqslant\eta}$ in a smooth function on $\mathbb{R}^+$. We define $S_b^\eta := S_b * \chi_\eta$ where χ_η is a mollifier on $\mathbb{R}^4$. Since S_b is locally Lipschitz (see Lemma 3.2.7), S_b^η is smooth and we get from Claim (1) of Lemma 3.2.7 and Claim (2) of Lemma 3.2.11 that
- $|S_b^\eta(x,y)| \leqslant C_b(1 + |x|^2 + |y|^2)$,
- $|\nabla_x S_b^\eta(x,y)|, |\nabla_y S_b^\eta(x,y)| \leqslant C_b(1 + |x| + |y|)$.

Since S_b is locally Lipschitz, S_b^η and ∇S_b^η converge locally uniformly to S_b and ∇S_b (see for example [14, Proposition 4.21]) and therefore we get the convergence of $g_b^\eta(x,y)$ and $\nabla g_b^\eta(x,y)$ to $g_b(x,y)$ and $\nabla g_b(x,y)$ for any $x \neq y$. □

With this regularisation we can compute the time derivative of T_1:

Claim 3.4.4. $T_1 \in W^{1,\infty}([0,T])$ and for almost every $t \in [0,T]$, we have

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = 2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(u(t,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)}\right) \cdot \nabla_x g_b(x,y)\omega(t,x)\omega(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Proof of Claim 3.4.4. For $0 \leqslant s,t \leqslant T$ and $0 < \eta < 1$ we have:

$$T_1(t) - T_1(s) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)(\omega(t,x)\omega(t,y) - \omega(s,x)\omega(s,y))\,\mathrm{d}x\,\mathrm{d}y.$$

Now for almost all x and y such that $x \neq y$,

$$\begin{aligned}|g_b^\eta(x,y)||\omega(t,x)\omega(t,y) - \omega(s,x)\omega(s,y)| \\ \leqslant C_b(|g(x-y)| + 1 + |x|^2 + |y|^2)|\omega(t,x)\omega(t,y) - \omega(s,x)\omega(s,y)|\end{aligned}$$

and

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (|g(x-y)| + 1 + |x|^2 + |y|^2) \\ \times |\omega(t,x)\omega(t,y) - \omega(s,x)\omega(s,y)|\,\mathrm{d}x\,\mathrm{d}y < +\infty\end{aligned}$$

because $\omega \in L^\infty$ with compact support. Therefore by dominated convergence theorem we get that

(3.4.1)
$$T_1(t) - T_1(s) = \lim_{\eta\to 0} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b^\eta(x,y)(\omega(t,x)\omega(t,y) - \omega(s,x)\omega(s,y))\,\mathrm{d}x\,\mathrm{d}y.$$

Since g_b^η is smooth and ω has compact support, we can use (3.1.3) to get that

$$\begin{aligned}\int_{\mathbb{R}^2} g_b^\eta(x,y)(\omega(t,y) - \omega(s,y))\,\mathrm{d}y = \\ \int_s^t \int_{\mathbb{R}^2} \nabla_y g_b^\eta(x,y) \cdot \left(u(\tau,y) - \alpha\frac{\nabla^\perp b(y)}{b(y)}\right)\omega(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau.\end{aligned}$$

Let us write

$$\varphi(t,x) := \int_{\mathbb{R}^2} g_b^\eta(x,y)\omega(t,y)\,\mathrm{d}y.$$

Since g_b^η is smooth we have that for any compact $K \subset \mathbb{R}^2$,

$$\begin{aligned}(t,x) \mapsto \\ \int_{\mathbb{R}^2} \nabla_y g_b^\eta(x,y) \cdot \left(u(t,y) - \alpha\frac{\nabla^\perp b(y)}{b(y)}\right)\omega(t,y)\,\mathrm{d}y \in L^\infty([0,T], \mathcal{C}^\infty(K))\end{aligned}$$

and thus $\varphi \in W^{1,\infty}([0,T], \mathcal{C}^\infty(K))$ and for almost every $t \in [0,T]$,

$$\partial_t \varphi(t,x) = \int_{\mathbb{R}^2} \nabla_y g_b^\eta(x,y) \left(u(\tau,y) - \alpha \frac{\nabla^\perp b(y)}{b(y)} \right) \omega(t,y) \, \mathrm{d}y.$$

Therefore we can use φ as a test function in (3.1.3) (remark that we defined (3.1.3) for smooth functions only but by density we can extend it to functions which are only $W^{1,\infty}$ in time) and we get that

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b^\eta(x,y)(\omega(t,x)\omega(t,y) - \omega(s,x)\omega(s,y)) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_y g_b^\eta(x,y) \cdot \left(u(\tau,y) - \alpha \frac{\nabla^\perp b(y)}{b(y)} \right) \omega(\tau,y)\omega(\tau,x) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + \int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_x g_b^\eta(x,y) \cdot \left(u(\tau,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)} \right) \omega(\tau,x)\omega(\tau,y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau.
\end{aligned}$$

Now we have that for almost every x and y such that $x \neq y$ and almost every $\tau \in [0,T]$,

$$\begin{aligned}
&|\nabla_x g_b^\eta(x,y) \cdot \left(u(\tau,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)} \right) \omega(\tau,x)\omega(\tau,y)| \\
&\leqslant C_b(|x-y|^{-1} + 1 + |x|^2 + |y|^2) \left| u(\tau,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)} \right| |\omega(\tau,y)||\omega(\tau,x)|
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} (|x-y|^{-1} + 1 + |x|^2 + |y|^2) \\
&\qquad\qquad \times \left| u(\tau,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)} \right| |\omega(\tau,y)||\omega(\tau,x)| \, \mathrm{d}x \, \mathrm{d}y < +\infty.
\end{aligned}$$

Applying dominated convergence theorem we find that

$$\begin{aligned}
&\int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_x g_b^\eta(x,y) \cdot \left(u(\tau,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)} \right) \omega(\tau,x)\omega(\tau,y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau \\
&\underset{\eta\to 0}{\longrightarrow} \int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_x g_b(x,y) \cdot \left(u(\tau,x) - \alpha \frac{\nabla^\perp b(x)}{b(x)} \right) \omega(\tau,x)\omega(\tau,y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\tau.
\end{aligned}$$

We can do the same for the first term to get that

$$\begin{aligned}
&\int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_y g_b^\eta(x,y) \cdot \left(u(\tau,y) - \alpha \frac{\nabla^\perp b(y)}{b(y)} \right) \omega(\tau,y)\omega(\tau,x) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau \\
&\underset{\eta\to 0}{\longrightarrow} \int_s^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_y g_b(x,y) \cdot \left(u(\tau,y) - \alpha \frac{\nabla^\perp b(y)}{b(y)} \right) \omega(\tau,y)\omega(\tau,x) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned}$$

Using that $\nabla_y g_b(x,y) = \nabla_x g_b(y,x)$ and (3.4.1) we get that $T_1 \in W^{1,\infty}([0,T])$ and for almost every $t \in [0,T]$, we get Claim 3.4.4. □

We know by Equation 3.3.5 that

$$\dot{E}_N = -\frac{2\alpha_N}{N^2}\sum_{i=1}^N \frac{\nabla^\perp b(q_i)}{b(q_i)} \cdot \sum_{\substack{j=1\\ j\neq i}}^N \nabla_x g_b(q_i, q_j)$$

and therefore

$$\dot{E}_N = -2\alpha_N \iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x,y)\,\mathrm{d}\omega_N(x)\,\mathrm{d}\omega_N(y). \tag{3.4.2}$$

Now we compute the derivative of the second term:

Claim 3.4.5. T_2 is Lipschitz and for almost every $t \in [0,T]$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&T_2(t)\\ &= -2\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(u(t,x) - \alpha\frac{\nabla^\perp b(x)}{b(x)}\right)\cdot\nabla_x g_b(x,y)\omega(t,x)\,\mathrm{d}x\,\mathrm{d}\omega_N(t,y)\\ &\quad + 2\alpha_N \iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\nabla^\perp b(x)}{b(x)}\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(x)\omega(t,y)\,\mathrm{d}y\\ &\quad + 2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(t,x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(x)\,\mathrm{d}\omega_N(y).\end{aligned}$$

Proof of Claim 3.4.5. If we use the regularisation g_b^η we defined in Claim 3.4.3, Equation (3.1.3) and if we let η tends to zero as we did for the proof of Claim 3.4.4, we can show that T_2 is Lipschitz and that for almost every $t \in [0,T]$, we have

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = T_{2,1} + T_{2,2}$$

where

$$\begin{aligned}T_{2,1} &:= -\frac{2}{N}\sum_{i=1}^N\int_{\mathbb{R}^2}\left(u(t,x) - \alpha\frac{\nabla^\perp b(x)}{b(x)}\right)\cdot\nabla_x g_b(x,q_i)\omega(t,x)\,\mathrm{d}x\\ &= -2\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(u(t,x) - \alpha\frac{\nabla^\perp b(x)}{b(x)}\right)\cdot\nabla_x g_b(x,y)\omega(t,x)\,\mathrm{d}x\,\mathrm{d}\omega_N(t,y)\end{aligned} \tag{3.4.3}$$

and

$$\begin{aligned}T_{2,2} &:= -\frac{2}{N}\sum_{i=1}^N \dot{q}_i\cdot\int_{\mathbb{R}^2}\nabla_y g_b(x,q_i)\omega(t,x)\,\mathrm{d}x\\ &= -\frac{2}{N}\sum_{i=1}^N \dot{q}_i\cdot\int_{\mathbb{R}^2}\nabla_x g_b(q_i,x)\omega(t,x)\,\mathrm{d}x\end{aligned}$$

$$=\left[\frac{2\alpha_N}{N}\sum_{i=1}^{N}\frac{\nabla^\perp b(q_i)}{b(q_i)}\cdot\int_{\mathbb{R}^2}\nabla_x g_b(q_i,x)\omega(t,x)\,\mathrm{d}x\right]$$
$$+\left[\frac{2}{N^2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{b(q_i)}\nabla_x^\perp g_b(q_i,q_j)\cdot\int_{\mathbb{R}^2}\nabla_x g_b(q_i,x)\omega(t,x)\,\mathrm{d}x\right]$$
$$=:T_{2,2,1}+T_{2,2,2}.$$

Now we have

$$\begin{aligned}T_{2,2,1}&=\frac{2\alpha_N}{N}\sum_{i=1}^{N}\frac{\nabla^\perp b(q_i)}{b(q_i)}\cdot\int_{\mathbb{R}^2}\nabla_x g_b(q_i,x)\omega(t,x)\,\mathrm{d}x\\&=2\alpha_N\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\nabla^\perp b(x)}{b(x)}\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(x)\omega(t,y)\,\mathrm{d}y\end{aligned}\tag{3.4.4}$$

and using $\int_{\mathbb{R}^2}\nabla_x g_b(q_i,y)\omega(y)\,\mathrm{d}y=b(q_i)u^\perp(t,q_i)$ (see Proposition 3.2.8), we get

$$\begin{aligned}T_{2,2,2}=&\frac{2}{N^2}\sum_{i=1}^{N}u(t,q_i)\cdot\sum_{\substack{j=1\\j\neq i}}^{N}\nabla_x^\perp g_b(q_i,q_j)\\=&2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}u(t,x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(x)\,\mathrm{d}\omega_N(y).\end{aligned}$$

Combining the upper equality with (3.4.3) and(3.4.4) we get the proof of Claim (3.4.5). ☐

Now remark that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2}u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(x)\,\mathrm{d}\omega(y)=\int_{\mathbb{R}^2}u\cdot bu^\perp\,\mathrm{d}\omega_N=0.$$

Thus combining Claim 3.4.4, Equation (3.4.2) and Claim 3.4.5 we obtain Proposition 3.4.1. ☐

We now compute the derivative of the rescaled modulated energy:

Proof of Proposition 3.4.2. We split $\overline{\mathcal{F}}_{b,N}$ in three terms:

$$\begin{aligned}\overline{\mathcal{F}}_{b,N}=&\iint_{\mathbb{R}^2\times\mathbb{R}^2}g_b(x,y)\overline{\omega}(t,x)\overline{\omega}(t,y)\,\mathrm{d}x\,\mathrm{d}y\\&-\frac{2}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^2}g_b(x,\overline{q_i})\overline{\omega}(t,x)\,\mathrm{d}x+\overline{E_N}\\=:&T_1+T_2+\overline{E_N}.\end{aligned}$$

Let us compute the time derivative of the first term. Using the regularisation g_b^η we defined in Claim 3.4.3 and using (3.1.3) and letting η tends to zero as we did for the proof of Claim 3.4.4, one can show that T_1 is Lipschitz and that for almost every $t \in [0, T]$, we have

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = -2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x, y) \overline{\omega}(t, x) \overline{\omega}(t, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{3.4.5}$$

For the derivative of $\overline{E_N}$ we rescale Equation (3.4.2) to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{E_N} = -2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x, y) \, \mathrm{d}\overline{\omega}_N(x) \, \mathrm{d}\overline{\omega}_N(y). \tag{3.4.6}$$

Now let us compute the derivative of the second term:

Claim 3.4.6.

$$\begin{aligned} \frac{\mathrm{d}T_2}{\mathrm{d}t} =& 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x, y) \overline{\omega}(t, x) \, \mathrm{d}x \, \mathrm{d}\overline{\omega}_N(t, y) \\ &+ 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x, y) \, \mathrm{d}\overline{\omega}_N(x) \, \mathrm{d}\overline{\omega}(y) \\ &+ \frac{2}{N^2 \alpha_N} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{v(t, \overline{q_i})}{b(\overline{q_i})} \cdot \nabla_x g_b(\overline{q_i}, \overline{q_j}). \end{aligned}$$

Proof of Claim 3.4.6. Using the regularisation g_b^η we defined in Claim 3.4.3 and using (3.1.3) and letting η tends to zero as we did for the proof of Claim 3.4.4, one can show that T_2 is Lipschitz and that for almost every $t \in [0, T]$, we have

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = T_{2,1} + T_{2,2}$$

where

$$\begin{aligned} T_{2,1} :=& \frac{2}{N} \sum_{i=1}^N \int_{\mathbb{R}^2} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x, q_i) \overline{\omega}(t, x) \, \mathrm{d}x \\ =& 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\nabla^\perp b(x)}{b(x)} \cdot \nabla_x g_b(x, y) \overline{\omega}(t, x) \, \mathrm{d}x \, \mathrm{d}\overline{\omega}_N(t, y) \end{aligned}$$

and

$$\begin{aligned} T_{2,2} :=& - \frac{2}{N} \sum_{i=1}^N \dot{\overline{q_i}} \cdot \int_{\mathbb{R}^2} \nabla_y g_b(x, q_i) \overline{\omega}(t, x) \, \mathrm{d}x \\ =& - \frac{2}{N} \sum_{i=1}^N \dot{\overline{q_i}} \cdot v(t, \overline{q_i}) \end{aligned}$$

$$
\begin{aligned}
=&\frac{2}{N}\sum_{i=1}^{N} v(t,\overline{q_i})\cdot\left[\frac{\nabla^\perp b(\overline{q_i})}{b(\overline{q_i})}+\frac{1}{N\alpha_N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{b(\overline{q_i})}\nabla_x g_b(\overline{q_i},\overline{q_j})\right]\\
=&2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\frac{\nabla^\perp b(x)}{b(x)}\cdot\nabla_x g_b(x,y)\,\mathrm{d}\overline{\omega}_N(x)\,\mathrm{d}\overline{\omega}(y)\\
&+\frac{2}{N^2\alpha_N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{v(t,\overline{q_i})}{b(\overline{q_i})}\cdot\nabla_x g_b(\overline{q_i},\overline{q_j})
\end{aligned}
$$

and thus we have Claim 3.4.6. ☐

Combining Equations (3.4.5), (3.4.6) and Claim 3.4.6 we get (3.4.2). ☐

3.5 Properties of the modulated energy

For $0<\eta<1$, we denote

$$
H_{N,\eta}:=G_b\left[\frac{1}{N}\sum_{i=1}^{N}\widetilde{\delta}_{q_i}^{(\eta)}-\omega\right].
$$

If $b=1$ this quantity is the electric potential introduced by Serfaty in [81, Equation (3.12)] divided by N.

Proposition 3.5.1. *Let $\omega\in\mathcal{P}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ with compact support and $q_1,\ldots,q_N\in\mathbb{R}^2$ be such that $q_i\neq q_j$ if $i\neq j$. Then the following inequality holds:*

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\frac{1}{b}|\nabla H_{N,\eta}|^2+\frac{C_b}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(g(q_i-q_j)-g^{(\eta)}(q_i-q_j))\\
&\leqslant\mathcal{F}_b(Q_N,\omega)+C_b\left(\frac{g(\eta)}{N}+I(Q_N)(\eta+N^{-1})+\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\,g(\eta)\eta\right)
\end{aligned}
$$

where $g^{(\eta)}$ is defined by (3.2.12).

From this proposition we see that even if it is not necessarily positive, the modulated energy is bounded from below by some negative power of N (provided that $(I(Q_N))$ is bounded). We will also prove the three following corollaries:

Corollary 3.5.2. *If ω and Q_N satisfy the hypothesis of Proposition 3.5.1, then there exists $c>0$ such that*

$$
\begin{aligned}
\frac{c}{N^2}|\{(q_i,q_j);|q_i-q_j|\leqslant\varepsilon\}|\leqslant&\mathcal{F}_b(Q_N,\omega)+C_b\bigg(\frac{g(\varepsilon)}{N}+I(Q_N)(\varepsilon+N^{-1})\\
&+\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\,g(\varepsilon)\varepsilon\bigg).
\end{aligned}
$$

Corollary 3.5.3. *Let $\alpha \in (0,1)$ and ξ be a test function (for example smooth with compact support or in the Schwartz space), then if ω and Q_N satisfy the hypothesis of Proposition 3.5.1 we have*

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \xi \left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right)\right| \leqslant & C_b|\xi|_{\mathcal{C}^{0,\alpha}} N^{-\alpha} + C_b\left(\int_{\mathbb{R}^2} \frac{1}{b}|\nabla\xi|^2\right)^{\frac{1}{2}} \Big(\mathcal{F}_b(\omega, Q_N) \\ & + \frac{\ln(N)}{N} + I(Q_N)N^{-1} \\ & + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} \frac{\ln(N)}{N}\Big)^{\frac{1}{2}}.\end{aligned}$$

In particular, there exists $\beta > 0$ such that for all $s < -1$,

$$\begin{aligned}\left\|\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right\|_{H^s} \leqslant & C_b((1 + I(Q_N) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty})N^{-\beta} \\ & + \mathcal{F}_b(\omega, Q_N)).\end{aligned}$$

Corollary 3.5.4. *If ω and Q_N satisfy the hypothesis of Proposition 3.5.1 and if $(I(Q_N))$ is bounded, then the two following assertions are equivalent:*

1. $\mathcal{F}_b(\omega, Q_N) \underset{N\to+\infty}{\longrightarrow} 0$.
2. $\displaystyle\frac{1}{N}\sum_{i=1}^N \delta_{q_i} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \omega$ *for the weak-$*$ topology of probability measures and*

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i, q_j) \longrightarrow \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Proposition 3.5.1 and Corollaries 3.5.2, 3.5.3 and 3.5.4 are analogues of other results obtained in [26, 71, 81]. Proposition 3.5.1 is an equivalent of [81, Proposition 3.3] or [71, Proposition 2.2] and the proof will follow the same steps: regularise the modulated energy and control the remainders. Some terms are very similar to the ones obtained in the Coulomb case whereas other terms are specific to the lake kernel and will be handled using the estimates proved in Section 3.2.

Corollary 3.5.2 is an equivalent of [71, Corollary 2.3] and Corollary 3.5.3 is an equivalent of [81, Proposition 3.6] (see Proposition 1.2.1). Both can be deduced from Proposition 3.5.1 in the same way [71, Corollary 2.3] and [81, Proposition 3.6] are deduced from [81, Proposition 3.3] or [71, Proposition 2.2].

Corollary 3.5.4 is an equivalent of [26, Lemma 2.6] (see Proposition 1.2.2) and its proof proceeds in the same way. Due to the bound we assumed on the moment of inertia, tightness issues will be easier to handle.

Let us begin by proving the main proposition of this section:

Proof of Proposition 3.5.1. Let us regularise the modulated energy (3.1.8) using the regularisation of the dirac mass $\widetilde{\delta}$ defined in (3.2.13). We have

$$\begin{aligned}
&\mathcal{F}_b(Q_N,\omega) = \\
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right)(x)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right)(y) \\
&+ \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(\sqrt{b(q_i)b(q_j)}g(q_i-q_j) \\
&- \sqrt{b(x)b(y)}g(x-y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y) \\
&+ \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(S_b(q_i,q_j) - S_b(x,y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y) \\
&- \frac{1}{N^2}\sum_{i=1}^N\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y) \\
&+ \frac{2}{N}\sum_{i=1}^N\iint_{\mathbb{R}^2\times\mathbb{R}^2}(\sqrt{b(x)b(y)}g(x-y) \\
&- \sqrt{b(x)b(q_i)}g(x-q_i))\omega(x)\,\mathrm{d}x\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y) \\
&+ \frac{2}{N}\sum_{i=1}^N\iint_{\mathbb{R}^2\times\mathbb{R}^2}(S_b(x,y) - S_b(x,q_i))\omega(x)\,\mathrm{d}x\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y) \\
=:&T_1+T_2+T_3+T_4+T_5+T_6.
\end{aligned}$$

Claim 3.5.5. We have

$$T_1 = \int_{\mathbb{R}^2}\frac{1}{b}|\nabla H_{N,\eta}|^2.$$

Proof of Claim 3.5.5. Let us first fix μ smooth with compact support and average zero and write $H_\mu = G_b[\mu]$. By Proposition 3.2.8, we have

$$\begin{aligned}
\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\mu(x)\mu(y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^2} H_\mu(x)\mu(x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^2} H_\mu(x)\,\mathrm{div}\left(\frac{1}{b}\nabla H_\mu\right)(x)\,\mathrm{d}x.
\end{aligned}$$

Let $R>0$, then integrating by parts we get

$$-\int_{B(0,R)} H_\mu\,\mathrm{div}\left(\frac{1}{b}\nabla H_\mu\right) = -\int_{\partial B(0,R)}\frac{1}{b}H_\mu\nabla H_\mu\cdot\mathrm{d}\vec{S} + \int_{B(0,R)}\frac{1}{b}|\nabla H_\mu|^2.$$

Using Proposition 3.2.5 applied to $\omega=\mu$, $u = -\frac{1}{b}\nabla^\perp H_\mu$ and $\psi = H_\mu$, we have

$$\left|\int_{\partial B(0,R)}\frac{1}{b}H_\mu\nabla H_\mu\cdot\mathrm{d}\vec{S}\right| \leqslant \frac{C}{R^2}(1+R^\delta)R \underset{R\to+\infty}{\longrightarrow} 0$$

and therefore

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\mu(x)\mu(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2}\frac{1}{b}|\nabla H_\mu|^2.$$

Now consider a sequence (μ_k) of smooth functions with compact support and average zero converging to $m := \frac{1}{N}\sum_{i=1}^{N}\widetilde{\delta}_{q_i}^{(\eta)} - \omega$ in $\dot{H}^{-1}$, then by Lemma 3.2.12,

$$\nabla H_{\mu_k} \underset{k\to+\infty}{\longrightarrow} \nabla H_{N,\eta} \text{ in } L^2.$$

and therefore

$$\int_{\mathbb{R}^2}\frac{1}{b}|\nabla H_{\mu_k}|^2 \underset{k\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2}\frac{1}{b}|\nabla H_{N,\eta}|^2$$

and

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\mu_k(x)\mu_k(y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\,\mathrm{d}m(x)\,\mathrm{d}m(y)\right| \\
&\quad= \left|\int_{\mathbb{R}^2} G_b[\mu_k - m]\,\mathrm{d}\mu_k + \int_{\mathbb{R}^2} G_b[m]\,\mathrm{d}(\mu_k - m)\right| \\
&\quad\leqslant C\,\|\nabla G_b[\mu_k - m]\|_{L^2}\,\|\mu_k\|_{\dot{H}^{-1}} + C\,\|\nabla G_b[m]\|_{L^2}\,\|\mu_k - m\|_{\dot{H}^{-1}} \\
&\quad\leqslant C\,\|\mu_k - m\|_{\dot{H}^{-1}}
\end{aligned}$$

by Lemma 3.2.12 so we get Claim 3.5.5. □

Now let us bound the fourth term:

Claim 3.5.6.

$$|T_4| \leqslant \frac{C_b}{N}(g(\eta) + I(Q_N)).$$

Proof of Claim 3.5.6. We write

$$\begin{aligned}
T_4 = &-\frac{1}{N^2}\sum_{i=1}^{N}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\sqrt{b(x)b(y)}g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y) \\
&-\frac{1}{N^2}\sum_{i=1}^{N}\iint_{\mathbb{R}^2\times\mathbb{R}^2} S_b(x,y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y) \\
=:&T_{4,1} + T_{4,2}.
\end{aligned}$$

Using the definition of $\widetilde{\delta}_q$ (3.2.13) and Lemma 2.3.1 we get

$$T_{4,1} = -\frac{1}{N^2}\sum_{i=1}^{N} m_b(q_i,\eta)^2\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)\,\mathrm{d}\delta_{q_i}^{(\eta)}(y)$$

$$= -\frac{1}{N^2}\sum_{i=1}^{N} m_b(q_i,\eta)^2 \int_{\mathbb{R}^2} g^{(\eta)}(x-q_i)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x).$$

Therefore,

$$|T_{4,1}| \leqslant \frac{C_b g(\eta)}{N}.$$

Now by Claim (2) of Lemma 3.2.11, we have

$$|T_{4,2}| \leqslant \frac{C_b}{N^2}\sum_{i=1}^{N}(1+|q_i|^2).$$

We get that

$$|T_4| \leqslant \frac{C_b}{N}(1+I(Q_N)+g(\eta)) \leqslant \frac{C_b}{N}(g(\eta)+I(Q_N)).$$

□

Now we bound the third and the sixth term:

Claim 3.5.7.

$$|T_3| + |T_6| \leqslant C_b(\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)} + I(Q_N))\eta.$$

Proof of Claim 3.5.7. For $x \in \partial B(q_i,\eta), y \in \partial B(q_j,\eta)$, we use Claim (1) of Lemma 3.2.7 and the symmetry of S_b to get

$$\begin{aligned}|S_b(q_i,q_j) - S_b(x,y)| &\leqslant |S_b(q_i,q_j) - S_b(x,q_j)| + |S_b(x,q_j) - S_b(x,y)| \\ &\leqslant C_b(1+|q_j|)\eta + C_b(1+|q_i|)\eta \\ &\leqslant C_b(1+|q_i|+|q_j|)\eta.\end{aligned}$$

Thus we can bound the third term:

$$|T_3| \leqslant C_b(1+I(Q_N))\eta. \tag{3.5.1}$$

The sixth term can be bounded in the same way:

$$|T_6| \leqslant \frac{C_b}{N}\sum_{i=1}^{N}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(1+|x|+|q_i|)\eta\omega(x)\,\mathrm{d}x\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y).$$

We get that

$$|T_6| \leqslant C_b(\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)} + I(Q_N))\eta. \tag{3.5.2}$$

and combining (3.5.1) with (3.5.2) we get Claim 3.5.7. □

Now let us bound the fifth term:

Claim 3.5.8.

$$|T_5| \leqslant C_b \|\omega\|_{L^1\cap L^\infty} \eta g(\eta).$$

Proof of Claim 3.5.8. Using Lemma 2.3.1 we write T_5 as

$$\begin{aligned}
T_5 =& \frac{2}{N}\sum_{i=1}^N \int_{\mathbb{R}^2} (m_b(q_i,\eta)g^{(\eta)}(x-q_i) - \sqrt{b(q_i)}g(x-q_i))\sqrt{b(x)}\omega(x)\,\mathrm{d}x \\
=& \frac{2}{N}\sum_{i=1}^N (m_b(q_i,\eta) - \sqrt{b(q_i)}) \int_{\mathbb{R}^2} g^{(\eta)}(x-q_i)\sqrt{b(x)}\omega(x)\,\mathrm{d}x \\
&+ \frac{2}{N}\sum_{i=1}^N \sqrt{b(q_i)} \int_{\mathbb{R}^2} (g^{(\eta)}(x-q_i) - g(x-q_i))\sqrt{b(x)}\omega(x)\,\mathrm{d}x.
\end{aligned}$$

and thus by (3.2.15) and since $|g^{(\eta)}(x-q_i)| \leqslant C(g(\eta)+|x|+|q_i|)$ we have

$$\begin{aligned}
|T_5| \leqslant& C_b \|\omega\|_{L^1} \eta g(\eta) + C_b \|\omega\|_{L^1(|x|\,\mathrm{d}x)} \eta + C_b \|\omega\|_{L^1} (1+I(Q_N))\eta \\
&+ C_b \|\omega\|_{L^\infty} \int_{B(0,\eta)} |g^{(\eta)}(x) - g(x)|\,\mathrm{d}x.
\end{aligned}$$

We get that

$$|T_5| \leqslant C_b \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} \eta g(\eta) + (1+I(Q_N))\eta$$

since ω is a probability density. □

We are only remained to estimate from below the second term:

Claim 3.5.9.

$$T_2 \geqslant \frac{C_b}{N^2} \sum_{1\leqslant i\neq j\leqslant N} (g(q_i-q_j) - g^{(\eta)}(q_i-q_j)) - C_b\eta g(\eta).$$

Proof of Claim 3.5.9. We also split T_2 in two terms:

$$\begin{aligned}
T_2 =& \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \sqrt{b(q_i)b(q_j)}g(q_i-q_j) \\
&- m_b(q_i,\eta)m_b(q_j,\eta) \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x-y)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)\,\mathrm{d}\delta_{q_j}^{(\eta)}(y) \\
=& \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \sqrt{b(q_i)b(q_j)}g(q_i-q_j) \\
&- m_b(q_i,\eta)m_b(q_j,\eta) \int_{\mathbb{R}^2} g^{(\eta)}(q_i-y)\,\mathrm{d}\delta_{q_j}^{(\eta)}(y) \\
=& \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} (\sqrt{b(q_i)b(q_j)} - m_b(q_i,\eta)m_b(q_j,\eta))
\end{aligned}$$

$$\times \int_{\mathbb{R}^2} g^{(\eta)}(q_i - y)\,\mathrm{d}\delta_{q_j}^{(\eta)}(y)$$
$$+ \frac{1}{N^2} \sum_{1\leqslant i\neq j\leqslant N} \sqrt{b(q_i)b(q_j)} \left(g(q_i - q_j) - \int_{\mathbb{R}^2} g^{(\eta)}(q_i - y)\,\mathrm{d}\delta_{q_j}^{(\eta)}(y) \right)$$
$$= T_{2,1} + T_{2,2}.$$

Writing

$$\sqrt{b(q_i)b(q_j)} - m_b(q_i,\eta)m_b(q_j,\eta) = \sqrt{b(q_i)}(\sqrt{b(q_j)} - m_b(q_j,\eta)) + m_b(q_j,\eta)(\sqrt{b(q_i)} - m_b(q_i,\eta))$$

and using (3.2.15) we get that

$$|T_{2,1}| \leqslant C_b \eta g(\eta). \tag{3.5.3}$$

Now by Lemma 2.3.1,

$$\begin{aligned} &g(q_i - q_j) - \int_{\mathbb{R}^2} g^{(\eta)}(q_i - y)\,\mathrm{d}\delta_{q_j}^{(\eta)}(y) \\ &= g(q_i - q_j) - g^{(\eta)}(q_i - q_j) + \int_{\mathbb{R}^2} (g(q_i - y) - g^{(\eta)}(q_i - y))\,\mathrm{d}\delta_{q_j}^{(\eta)}(y) \\ &\geqslant g(q_i - q_j) - g^{(\eta)}(q_i - q_j) + 0 \end{aligned}$$

and thus

$$T_{2,2} \geqslant \frac{C_b}{N^2} \sum_{1\leqslant i\neq j\leqslant N} (g(q_i - q_j) - g^{(\eta)}(q_i - q_j)). \tag{3.5.4}$$

We get Claim 3.5.9 combining Equations (3.5.3) with (3.5.4). □

Combining Claims 3.5.5, 3.5.6, 3.5.7, 3.5.8 and 3.5.9 we get the proof of Proposition 3.5.1. □

Now we prove the "counting close particles" Corollary:

Proof of Corollary 3.5.2. The proof is exactly the same as the proof of [75, Lemma 3.7]. If $|q_i - q_j| \leqslant \varepsilon$ then

$$\begin{aligned} g(q_i - q_j) - g^{(2\varepsilon)}(q_i - q_j) &= -\frac{1}{2\pi} \ln|q_i - q_j| + \frac{1}{2\pi}\ln(2\varepsilon) \\ &\geqslant -\frac{1}{2\pi}\ln(\varepsilon) + \frac{1}{2\pi}\ln(2\varepsilon) = \frac{1}{2\pi}\ln(2) > 0. \end{aligned}$$

Thus, since $g - g^{(2\varepsilon)} \geqslant 0$,

$$\frac{1}{2\pi N^2} \ln(2) |\{(q_i, q_j); |q_i - q_j| \leqslant \varepsilon\}|$$

$$\begin{aligned}
&\leqslant \frac{1}{N^2} \sum_{\substack{1\leqslant i\neq j\leqslant N \\ |q_i-q_j|\leqslant\varepsilon}} (g(q_i-q_j)-g^{(2\varepsilon)}(q_i-q_j)) \\
&\leqslant \frac{1}{N^2} \sum_{1\leqslant i\neq j\leqslant N} (g(q_i-q_j)-g^{(2\varepsilon)}(q_i-q_j)) \\
&\leqslant \mathcal{F}_b(Q_N,\omega) \\
&\quad + C_b\bigg(\frac{g(\varepsilon)}{N} + I(Q_N)(\varepsilon+N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\, g(\varepsilon)\varepsilon\bigg).
\end{aligned}$$

where we used Proposition 3.5.1 in the last inequality. □

Now we prove the coercivity result:

Proof of Corollary 3.5.3. We have

$$\begin{aligned}
\int_{\mathbb{R}^2} \xi\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right) &= \frac{1}{N}\int_{\mathbb{R}^2} \xi\left(\sum_{i=1}^N \delta_{q_i} - \widetilde{\delta}_{q_i}^{(\eta)}\right) \\
&\quad + \int_{\mathbb{R}^2} \xi\left(\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right) \\
&=: T_1 + T_2.
\end{aligned}$$

Now,

$$\begin{aligned}
T_1 &= \frac{1}{N}\sum_{i=1}^N \xi(q_i) - m_b(q_i,\eta)\int_{\partial B(q_i,\eta)} \frac{\xi(x)}{\sqrt{b(x)}}\,\mathrm{d}\delta_{q_i}^{(\eta)}(x) \\
&= \frac{1}{N}\sum_{i=1}^N m_b(q_i,\eta)\int_{\partial B(q_i,\eta)} \frac{\xi(q_i)-\xi(x)}{\sqrt{b(x)}}\,\mathrm{d}\delta_{q_i}^{(\eta)}(x).
\end{aligned}$$

Thus

$$|T_1| \leqslant C_b|\xi|_{\mathcal{C}^{0,\alpha}}\eta^\alpha.$$

Using a sequence (μ_k) of smooth functions with compact support and average 0 converging to $\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega$ as we have done for Claim 3.5.5 we can show that

$$T_2 = \int_{\mathbb{R}^2} \frac{1}{b}\nabla\xi\cdot\nabla H_{N,\eta}$$

and therefore

$$|T_2| \leqslant \left(\int_{\mathbb{R}^2}\frac{1}{b}|\nabla\xi|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\frac{1}{b}|\nabla H_{N,\eta}|^2\right)^{\frac{1}{2}}$$

$$\leqslant C_b \left(\int_{\mathbb{R}^2} \frac{1}{b}|\nabla\xi|^2\right)^{\frac{1}{2}} \left(\mathcal{F}_b(Q_N,\omega) + \frac{g(\eta)}{N}\right.$$
$$\left. + I(Q_N)(\eta + N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} g(\eta)\eta\right)^{\frac{1}{2}}.$$

by Proposition 3.5.1. We conclude by taking $\eta = N^{-1}$. The bound on

$$\left\|\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right\|_{H^s}$$

follows from Sobolev embeddings. □

We finish this section by proving the weak-$*$ convergence result:

Proof of Corollary 3.5.4. Let us denote $\omega_N = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{q_i}$ and prove that (ω_N) is a tight sequence of probability measures. Let $R > 1$, then

$$\begin{aligned} |\{i \in [1,N]\ ;\ |q_i| \geqslant R\}|R^2 &\leqslant \sum_{\substack{i=1 \\ |q_i|\geqslant R}}^N |q_i|^2 \\ &\leqslant NI(Q_N). \end{aligned} \tag{3.5.5}$$

Dividing by NR^2 both sides of the inequality we get

$$\int_{B(0,R)^c} \mathrm{d}\omega_N \leqslant I(Q_N)R^{-2}$$

and since $(I(Q_N))$ is bounded we get that (ω_N) is tight. We will now prove the following Claim:

Claim 3.5.10. Assume that (ω_N) converges to ω for the weak-$*$ topology of probability measures and that $(I(Q_N))$ is bounded. Then $\mathcal{F}_b(Q_N,\omega) \underset{N\to+\infty}{\longrightarrow} 0$ if and only if we have

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i,q_j) \longrightarrow \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Proof of Claim 3.5.10. Let $\varepsilon > 0$. We write the modulated energy as the sum of three terms:

$$\begin{aligned} \mathcal{F}_b(Q_N,\omega) = &- \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i,q_j) \\ &- 2\int_{\mathbb{R}^2} \psi(y)\,\mathrm{d}(\omega_N-\omega)(y) \end{aligned} \tag{3.5.6}$$

where $\psi = G_b[\omega]$. Let $R \geqslant 1$ be such that $\mathrm{supp}(\omega) \subset B(0,R)$ and χ_R be a smooth function such that $0 \leqslant \chi \leqslant 1$, $\chi_R(x) = 1$ if $|x| \leqslant R$ and $\chi_R(x) = 0$ if $|x| \geqslant 2R$. We write

$$\begin{aligned}\int_{\mathbb{R}^2} \psi \,\mathrm{d}(\omega - \omega_N) &= \int \psi(1-\chi_R)\,\mathrm{d}(\omega-\omega_N) + \int \psi\chi_R\,\mathrm{d}(\omega-\omega_N) \\ &= -\int_{B(0,R)^c} \psi(1-\chi_R)\,\mathrm{d}\omega_N + \int \psi\chi_R\,\mathrm{d}(\omega-\omega_N).\end{aligned}$$

We bound the first term as we did to obtain (3.5.5):

$$\begin{aligned}\left|\int_{B(0,R)^c} \psi(1-\chi_R)\,\mathrm{d}\omega_N\right| &\leqslant \frac{1}{N}\sum_{\substack{i=1\\|q_i|\geqslant R}}^N |\psi(q_i)| \\ &\leqslant \frac{C_b}{N}\sum_{\substack{i=1\\|q_i|\geqslant R}}^N (1+|q_i|^\delta) \\ &\leqslant C_b(R^{-2}I(Q_N) + R^{-2+\delta}I(Q_N))\end{aligned}$$

for some $0 < \delta < 1$ (by Proposition 3.2.5). Therefore,

$$\left|\int \psi(1-\chi_R)\,\mathrm{d}\omega_N\right| \leqslant \varepsilon$$

if R is big enough. Now $\psi\chi_R$ is continuous and bounded (see Lemma 3.2.3), so by weak-$*$ convergence of (ω_N) to ω we get that

$$\int \psi\chi_R\,\mathrm{d}(\omega-\omega_N) \underset{N\to+\infty}{\longrightarrow} 0$$

and therefore

$$\limsup_{N\to+\infty}\left|\int_{\mathbb{R}^2}\psi\,\mathrm{d}(\omega-\omega_N)\right| \leqslant \varepsilon.$$

for all $\varepsilon > 0$, so we get

$$\int_{\mathbb{R}^2}\psi\,\mathrm{d}(\omega-\omega_N) \underset{N\to+\infty}{\longrightarrow} 0.$$

Using (3.5.6) we get that $\mathcal{F}_b(Q_N,\omega) \underset{N\to+\infty}{\longrightarrow} 0$ if and only if we have

$$\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} g_b(q_i,q_j) \longrightarrow \iint_{\mathbb{R}^2\times\mathbb{R}^2} g_b(x,y)\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y.$$

□

It follows directly from Claim 3.5.10 that (2) $\implies$ (1). Now if we have (1), using Corollary 3.5.3 we have convergence of (ω_N) to ω in any H^s for any $s < -1$. It follows by Prokhorov's theorem (ω_N) converges to ω for the weak-$*$ topology of probability measures. By Claim 3.5.10 we also have convergence of the interaction energy and therefore (1) $\implies$ (2). □

3.6 Proof of the main Proposition 3.6.1

Let us recall that for $q \in \mathbb{R}^2$, $Q_N = (q_1, ..., q_N) \in (\mathbb{R}^2)^N$ and $0 < \eta < 1$, we have denoted

$$I(Q_N) = \frac{1}{N}\sum_{i=1}^{N} |q_i|^2,$$

$$\widetilde{\delta}_q^{(\eta)} = m_b(q,\eta)\frac{\mathrm{d}\delta_q^{(\eta)}}{\sqrt{b}}$$

and

$$m_b(q,\eta) = \left(\int_{\mathbb{R}^2} \frac{\mathrm{d}\delta_q^{(\eta)}}{\sqrt{b}}\right)^{-1}$$

where $\delta_q^{(\eta)}$ is the uniform probability measure on the circle $\partial B(q,\eta)$.

In this Section, we prove the following result:

Proposition 3.6.1. *Let $Q_N = (q_1, ..., q_N) \in (\mathbb{R}^2)^N$ such that $q_i \neq q_j$ if $i \neq j$, $u \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ and $\omega \in \mathcal{P}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with compact support such that $\nabla G_b[\omega]$ is continuous and bounded. There exists $\beta \in (0,1)$ (independent of ω, u and Q_N) such that*

$$\begin{aligned}\Bigg| \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} & u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{q_i} - \omega\right)^{\otimes 2}(x,y)\Bigg| \\ &\leqslant C_b \|u\|_{W^{1,\infty}} |\mathcal{F}_b(Q_N,\omega)| \\ &\quad + C_b(1+\|u\|_{W^{1,\infty}})\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}(1+I(Q_N))N^{-\beta}.\end{aligned}$$

This proposition is an equivalent of [81, Proposition 1.1] (see Proposition 2.1.15) or [71, Proposition 4.1] and the proof will follow the same steps: regularise the dirac masses, use the structure of the lake kernel to bound the regular part and control the remainders. Some terms are very similar to the ones obtained in the Coulomb case and we will use both the properties of our regularisation (see Subsection 3.2.4) and some estimates proved in [71] to bound them. As in the proof of Proposition 3.5.1 some terms are specific to the lake kernel and we will use results of Section 3.2 to bound them.

Proof. Let us fix $0 < \eta < \dfrac{1}{8}$ and write

$$
\begin{aligned}
&\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right)^{\otimes 2}(x,y)\\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right)^{\otimes 2}(x,y)\\
&\quad + \Bigg(-\frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\Big[\mathrm{d}\omega(x)\,\mathrm{d}(\delta_{q_i}-\widetilde{\delta}_{q_i}^{(\eta)})(y)\\
&\quad + \mathrm{d}(\delta_{q_i}-\widetilde{\delta}_{q_i}^{(\eta)})(x)\,\mathrm{d}\omega(y)\Big]\Bigg)\\
&\quad + \Bigg(\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(x)\cdot\nabla_x g_b(x,y)\\
&\quad [\,\mathrm{d}\delta_{q_i}(x)\,\mathrm{d}\delta_{q_j}(y) - \mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y)]\Bigg)\\
&=: T_1 + T_2 + T_3.
\end{aligned}
\tag{3.6.1}
$$

Let us bound the first term. As in Section 3.5 we write

$$
H_{N,\eta} := G_b\left[\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right].
$$

We claim:

Claim 3.6.2.

$$
\begin{aligned}
T_1 = &-\int_{\mathbb{R}^2} u(x)\cdot\nabla H_{N,\eta}(x)\nabla\left(\frac{1}{b}\right)\cdot\nabla H_{N,\eta}(x)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^2}\nabla\left(\frac{1}{2b}u\right) : [H_{N,\eta}, H_{N,\eta}]
\end{aligned}
$$

Proof of Claim 3.6.2. This claim is similar to [81, Lemma 4.3] and we proceed the same way: Let us first fix μ smooth with compact support and average zero and write $H_\mu = G_b[\mu]$. Then

$$
\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\mu^{\otimes 2}(x,y)\\
&\qquad = -\int_{\mathbb{R}^2} u(x)\cdot\nabla H_\mu(x)\,\mathrm{div}\left(\frac{1}{b}\nabla H_\mu\right)(x)\,\mathrm{d}x\\
&\qquad = -\int_{\mathbb{R}^2} u(x)\cdot\nabla H_\mu(x)\nabla\left(\frac{1}{b}\right)\cdot\nabla H_\mu(x)\,\mathrm{d}x\\
&\qquad\quad -\int_{\mathbb{R}^2}\frac{1}{b}u\cdot\nabla H_\mu\Delta H_\mu.
\end{aligned}
$$

For the second integral of the right handside we proceed as in [81] and use the stress-energy tensor defined by (3.1.7) (for more details, see [81, Equality (1.25)] and the associated references):

$$\int_{\mathbb{R}^2} \frac{1}{b} u \cdot \nabla H_\mu \Delta H_\mu = \int_{\mathbb{R}^2} \frac{1}{2b} u \cdot \operatorname{div}([H_\mu, H_\mu]).$$

Integrating over a ball of radius R and integrating by parts we get

$$\begin{aligned}\int_{B(0,R)} \frac{1}{2b} u \cdot \operatorname{div}([H_\mu, H_\mu]) = &\int_{\partial B(0,R)} \frac{1}{2b}[H_\mu, H_\mu] u \cdot \mathrm{d}\vec{S} \\ &- \int_{B(0,R)} \nabla \left(\frac{1}{2b} u\right) : [H_\mu, H_\mu].\end{aligned}$$

Using Proposition 3.2.5 (applied to $\omega = \mu$ and $\psi = H_\mu$) we have

$$\left| \int_{\partial B(0,R)} \frac{1}{2b}[H_\mu, H_\mu] u \cdot \mathrm{d}\vec{S} \right| \leqslant \frac{C_{b,\mu} \|u\|_{L^\infty}}{R^4} R.$$

Letting $R \longrightarrow \infty$ we obtain

$$\int_{\mathbb{R}^2} \frac{1}{2b} u \cdot \operatorname{div}([H_\mu, H_\mu]) = -\int_{\mathbb{R}^2} \nabla \left(\frac{1}{2b} u\right) : [H_\mu, H_\mu].$$

Now if (μ_k) is a sequence of smooth functions with compact support and average zero such that

$$\mu_k - \left(\frac{1}{N} \sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right) \underset{N \to +\infty}{\longrightarrow} 0 \text{ in } \dot{H}^{-1}$$

then by Lemma 3.2.12 we have

$$\nabla H_{\mu_k} \underset{N \to +\infty}{\longrightarrow} \nabla H_{N,\eta} \text{ in } L^2$$

and therefore since $u \in W^{1,\infty}$ and since $[H_{\mu_k}, H_{\mu_k}]$ (defined by Equation (3.1.7)) is quadratic in the derivatives of H_{μ_k} we get that

$$-\int_{\mathbb{R}^2} u(x) \cdot \nabla H_{\mu_k}(x) \nabla \left(\frac{1}{b}\right) \cdot \nabla H_{\mu_k}(x) \, \mathrm{d}x + \int_{\mathbb{R}^2} \nabla \left(\frac{1}{2b} u\right) : [H_{\mu_k}, H_{\mu_k}]$$

converges to

$$-\int_{\mathbb{R}^2} u(x) \cdot \nabla H_{N,\eta}(x) \nabla \left(\frac{1}{b}\right) \cdot \nabla H_{N,\eta}(x) \, \mathrm{d}x + \int_{\mathbb{R}^2} \nabla \left(\frac{1}{2b} u\right) : [H_{N,\eta}, H_{N,\eta}]$$

as $k \longrightarrow +\infty$. We are only left to justify that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\mu_k^{\otimes 2}(x,y)$$
$$\underset{k\to+\infty}{\longrightarrow} \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)} - \omega\right)^{\otimes 2}(x,y).$$

We define

$$m = \frac{1}{N}\sum_{i=1}^N \widetilde{\delta}_{q_i}^{(\eta)}.$$

Let us consider a sequence (ν_k) of smooth probability densities with support included in a ball $B(0,R)$ independent of k (containing $\mathrm{supp}(m)$), such that $(\nu_k - m)$ converges to zero in $\dot{H}^{-1}$ and for the weak-$*$ topology of probability measures. If we set $\mu_k = \nu_k - \omega$, then

$$\mu_k - (m-\omega) \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \dot{H}^{-1}.$$

Now we write

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\mu_k^{\otimes 2}(x,y) \\
&\quad - \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}(m-\omega)^{\otimes 2}(x,y) \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\omega(x)\,\mathrm{d}x\,\mathrm{d}(m-\nu_k)(y) \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\omega(y)\,\mathrm{d}y\,\mathrm{d}(m-\nu_k)(x) \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}(\nu_k\otimes\nu_k - m\otimes m)(x,y) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

We have

$$\begin{aligned}
|I_1| = \left|\int_{\mathbb{R}^2} u\cdot\nabla G_b[m-\nu_k]\omega\right| &\leqslant \|u\|_{L^\infty}\|\omega\|_{L^2}\|\nabla G_b[m-\nu_k]\|_{L^2} \\
&\leqslant C\|u\|_{L^\infty}\|\omega\|_{L^2}\|m-\nu_k\|_{\dot{H}^{-1}}
\end{aligned}$$

by Lemma 3.2.12 and therefore $I_1 \underset{k\to+\infty}{\longrightarrow} 0$. Recall that $(m-\nu_k)$ converges to zero for the weak-$*$ topology of probability measures. Therefore

$$I_2 = \int_{\mathbb{R}^2} u\cdot\nabla G_b[\omega]\,\mathrm{d}(m-\nu_k) \underset{k\to+\infty}{\longrightarrow} 0$$

since u and $\nabla G_b[\omega]$ are continuous and bounded by assumption. Now we want to show that I_3 converges to zero. Remark that writing $\mu_k = \nu_k - \omega$

and proving that I_1 and I_2 converge to zero allowed us to restrict ourself to study the convergence of

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\nu_k(x)\,\mathrm{d}\nu_k(y)$$

for ν_k nonnegative (which will be crucial for using Delort's argument below). We use the definition of g_b (3.2.10) to write

$$\begin{aligned}
I_3 =& \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla\sqrt{b}(x)\sqrt{b(y)}g(x-y)\,\mathrm{d}(\nu_k\otimes\nu_k - m\otimes m)(x,y)\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)b(y)}u(x)\cdot\nabla g(x-y)\,\mathrm{d}(\nu_k\otimes\nu_k - m\otimes m)(x,y)\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x S_b(x,y)\,\mathrm{d}(\nu_k\otimes\nu_k - m\otimes m)(x,y)\\
=:& I_{3,1}+I_{3,2}+I_{3,3}.
\end{aligned}$$

We write

$$\begin{aligned}
I_{3,1} =& \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla\sqrt{b}(x)\sqrt{b(y)}g(x-y)\,\mathrm{d}(\nu_k - m)(x)\,\mathrm{d}\nu_k(y)\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla\sqrt{b}(x)\sqrt{b(y)}g(x-y)\,\mathrm{d}m(x)\,\mathrm{d}(\nu_k - m)(y)\\
=& \int_{\mathbb{R}^2} (u\cdot\nabla\sqrt{b})(g*[\sqrt{b}\nu_k])\,\mathrm{d}(\nu_k - m)\\
&+\int_{\mathbb{R}^2} (u\cdot\nabla\sqrt{b})(g*[\sqrt{b}(\nu_k - m)])\,\mathrm{d}m.
\end{aligned} \tag{3.6.2}$$

Recall that $B(0,R)$ is a ball containing the supports of m and ν_k. Consider a smooth probability density ρ with support in $B(0,R)$. We define

$$\begin{aligned}
\chi_k &= \left(\int_{\mathbb{R}^2}\sqrt{b}\nu_k\right)\rho,\\
\chi_\infty &= \left(\int_{\mathbb{R}^2}\sqrt{b}\,\mathrm{d}m\right)\rho
\end{aligned}$$

and write

$$\begin{aligned}
\nabla g*(\sqrt{b}(\nu_k - m)) =& \nabla g*(\sqrt{b}\nu_k - \chi_k + \chi_\infty - \sqrt{b}m)\\
&+\left(\int_{\mathbb{R}^2}\sqrt{b}\nu_k - \int_{\mathbb{R}^2}\sqrt{b}\,\mathrm{d}m\right)\nabla g*\rho.
\end{aligned} \tag{3.6.3}$$

Now

$$\left\|\nabla g*(\sqrt{b}\nu_k - \chi_k + \chi_\infty - \sqrt{b}m)\right\|_{L^2}^2$$

$$= C \int_{\mathbb{R}^2} \frac{1}{|\xi|^2} |\widehat{\sqrt{b}\nu_k}(\xi) - \widehat{\chi_k}(\xi) + \widehat{\chi_\infty}(\xi) - \widehat{\sqrt{b}m}(\xi)|^2 \,\mathrm{d}\xi.$$

Remark that $\alpha_k = \sqrt{b}\nu_k - \chi_k + \chi_\infty - \sqrt{b}m$ is a Radon measure with support included in $B(0,R)$ such that $\widehat{\alpha_k}(0) = 0$. Therefore

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} e^{-ix\cdot\xi} \,\mathrm{d}\alpha_k(x) \right| &= \left| \int_{\mathbb{R}^2} (e^{-ix\cdot\xi} - 1) \,\mathrm{d}\alpha_k(x) \right| \\
&= 2 \left| \int_{\mathbb{R}^2} \sin\left(\frac{x\cdot\xi}{2}\right) \,\mathrm{d}\alpha_k(x) \right| \\
&\leqslant CR|\xi| \int_{\mathbb{R}^2} \mathrm{d}|\alpha_k|(x) \\
&\leqslant C_{b,R}|\xi|.
\end{aligned}$$

It follows that for $\varepsilon > 0$

$$\int_{|\xi|\leqslant\varepsilon} \frac{1}{|\xi|^2} |\widehat{\alpha_k}(\xi)|^2 \,\mathrm{d}\xi \leqslant C_{b,R}\varepsilon^2.$$

Moreover,

$$\begin{aligned}
&\int_{|\xi|\geqslant\varepsilon} \frac{1}{|\xi|^2} |\widehat{\sqrt{b}\nu_k}(\xi) - \widehat{\chi_k}(\xi) + \widehat{\chi_\infty}(\xi) - \widehat{\sqrt{b}m}(\xi)|^2 \,\mathrm{d}\xi \\
&\qquad \leqslant C_\varepsilon \left(\int_{\mathbb{R}^2} |\widehat{\chi_k}(\xi) - \widehat{\chi_\infty}(\xi)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}^2} \frac{1}{1+|\xi|^2} |\widehat{\sqrt{b}\nu_k}(\xi) - \widehat{\sqrt{b}m}(\xi)|^2 \right) \\
&\qquad \leqslant C_\varepsilon \left(\|\chi_k - \chi_\infty\|_{L^2} + \left\| \sqrt{b}\nu_k - \sqrt{b}m \right\|_{H^{-1}} \right) \underset{k\to+\infty}{\longrightarrow} 0
\end{aligned}$$

since b is smooth. Therefore

$$\limsup_{k\to+\infty} \left\| \nabla g * (\sqrt{b}\nu_k - \chi_k + \chi_\infty - \sqrt{b}m) \right\|_{L^2}^2 \leqslant C_{b,R}\varepsilon^2 \tag{3.6.4}$$

for all $\varepsilon > 0$ so

$$\nabla g * (\sqrt{b}\nu_k - \chi_k + \chi_\infty - \sqrt{b}m) \underset{k\to+\infty}{\overset{L^2}{\longrightarrow}} 0. \tag{3.6.5}$$

By Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]), $\nabla g * \rho \in L^p$ for all $2 < p < +\infty$ so

$$\left(\int_{\mathbb{R}^2} \sqrt{b}\nu_k - \int_{\mathbb{R}^2} \sqrt{b}m \right) \nabla g * \rho \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } L^2(B(0,R)).$$

Combining the upper limit with (3.6.3) and (3.6.5) we get that

$$\nabla g * (\sqrt{b}\nu_k) \underset{k\to+\infty}{\longrightarrow} \nabla g * (\sqrt{b}m) \text{ in } L^2(B(0,R)).$$

Now, by convolution inequality, we have

$$\left\| g * [\sqrt{b}\nu_k] \right\|_{L^2(B(0,R))} \leqslant C_b \|g\|_{L^2(B(0,2R))} \|\nu_k\|_{L^1} \leqslant C_b \|g\|_{L^2(B(0,2R))}$$

so $(g * [\sqrt{b}\nu_k])$ is bounded in $H^1(B(0,R))$ which is compactly embedded in $L^2(B(0,R))$. Therefore by (3.6.5), up to extraction, $(g*[\sqrt{b}\nu_k])$ converges to $g*[\sqrt{b}m]+C$ where C is a constant. If $x_0 \in B(0,R)$ is at a positive distance from the supports of ν_k and m then $g(x_0 - \cdot)$ is continuous on the supports of ν_k and m and therefore

$$g * [\sqrt{b}\nu_k](x_0) \underset{k\to+\infty}{\longrightarrow} g * [\sqrt{b}m](x_0)$$

by dominated convergence theorem. It follows that $C = 0$, thus

$$g * [\sqrt{b}\nu_k] \underset{k\to+\infty}{\longrightarrow} g * [\sqrt{b}m] \text{ in } H^1(B(0,R)).$$

We recall that since b is smooth,

$$\sqrt{b}\nu_k \underset{k\to+\infty}{\longrightarrow} \sqrt{b}m \text{ in } H^{-1}.$$

Moreover, $m \in H^{-1}$ with compact support and $u\cdot\nabla\sqrt{b} \in W^{1,\infty}$ so it follows by Decomposition 3.6.2 that

$$I_{3,1} \underset{k\to+\infty}{\longrightarrow} 0.$$

Since ∇g is antisymmetric we can write

$$\begin{aligned} I_{3,2} =& \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} H_u(x,y)\,\mathrm{d}(\sqrt{b}\nu_k)(x)\,\mathrm{d}(\sqrt{b}\nu_k)(y) \\ &- \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} H_u(x,y)\,\mathrm{d}(\sqrt{b}m)(x)\,\mathrm{d}(\sqrt{b}m)(y) \end{aligned}$$

where

$$H_u(x,y) = \frac{1}{2}(u(x)-u(y))\cdot\nabla g(x-y).$$

We recall that $(\sqrt{b}\nu_k)$ is a sequence of nonnegative functions with supports in $B(0,R)$ converging to $\sqrt{b}m$ in H^{-1} and for the weak-$*$ topology of measures with finite mass. Moreover, since u is Lipschitz, H_u is continuous outside of the diagonal and bounded. Therefore we can use Delort's argument (see [23, Proposition 1.2.6] or [78, Inequalities (3.4) and (3.5)]) to prove that

$$I_{3,2} \underset{k\to+\infty}{\longrightarrow} 0.$$

Finally we write

$$I_{3,3} = \int_{\mathbb{R}^2} u(x) \cdot \left(\int_{\mathbb{R}^2} \nabla_x S_b(x,y) \, \mathrm{d}\nu_k(y) \right) \, \mathrm{d}(\nu_k - m)(x) + \int_{\mathbb{R}^2} u(x) \cdot \left(\int_{\mathbb{R}^2} \nabla_x S_b(x,y) \, \mathrm{d}m(x) \right) \, \mathrm{d}(\nu_k - m)(y).$$

By Proposition 3.2.7 $u(x) \cdot \nabla_x S_b(x,y)$ is locally Hölder with respect to both variables and therefore since $\nu_k \otimes \nu_k - m \otimes m$ has compact support we have that $I_{3,3} \underset{k \to +\infty}{\longrightarrow} 0$. □

It follows from Claim 3.6.2 that

$$|T_1| \leqslant C_b \, \|u\|_{W^{1,\infty}} \int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2.$$

Hence by Proposition 3.5.1 we get

$$|T_1| \leqslant C_b \, \|u\|_{W^{1,\infty}} \Big(|\mathcal{F}_b(Q_N, \omega)| + \frac{g(\eta)}{N} + I(Q_N)(\eta + N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x) \cap L^\infty} \, g(\eta)\eta \Big). \tag{3.6.6}$$

Now let us split T_2 in three parts:

$$\begin{aligned} T_2 = & -\frac{1}{N} \sum_{i=1}^N \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \big(u(x) \cdot \nabla_x g_b(x,y) \\ & + u(y) \cdot \nabla_x g_b(y,x) \big) \omega(x) \, \mathrm{d}x \, \mathrm{d}(\delta_{q_i} - \widetilde{\delta}_{q_i}^{(\eta)})(y) \\ = & -\frac{1}{N} \sum_{i=1}^N \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \Big(u(x) \cdot \nabla \sqrt{b}(x) \sqrt{b(y)} \\ & + u(y) \cdot \nabla \sqrt{b}(y) \sqrt{b(x)} \Big) g(x-y) \omega(x) \, \mathrm{d}x \, \mathrm{d}(\delta_{q_i} - \widetilde{\delta}_{q_i}^{(\eta)})(y) \\ & - \frac{1}{N} \sum_{i=1}^N \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \sqrt{b(x)b(y)} (u(x) - u(y)) \\ & \cdot \nabla g(x-y) \omega(x) \, \mathrm{d}x \, \mathrm{d}(\delta_{q_i} - \widetilde{\delta}_{q_i}^{(\eta)})(y) \\ & - \frac{1}{N} \sum_{i=1}^N \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \Big(u(x) \cdot \nabla_x S_b(x,y) \\ & + u(y) \cdot \nabla_x S_b(y,x) \Big) \omega(x) \, \mathrm{d}x \, \mathrm{d}(\delta_{q_i} - \widetilde{\delta}_{q_i}^{(\eta)})(y) \\ =: & -(T_{2,1} + T_{2,2} + T_{2,3}). \end{aligned}$$

We will bound the three terms separately:

Claim 3.6.3. There exists $0 < s < 1$ such that

$$|T_{2,1}| \leqslant C_b \|u\|_{W^{1,\infty}} \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} (1 + I(Q_N))\eta^s.$$

Proof of Claim 3.6.3. Since $\widetilde{\delta}_{q_i}^{(\eta)}$ is a probability measure, we can write

$$\begin{aligned}
T_{2,1} =& \frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla\sqrt{b}(x)\cdot u(x)\omega(x)(\sqrt{b(q_i)}g(x-q_i) \\
&- \sqrt{b(y)}g(x-y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \\
&+ \frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)}\omega(x)(\nabla\sqrt{b}(q_i)\cdot u(q_i)g(x-q_i) \\
&- \nabla\sqrt{b}(y)\cdot u(y)g(x-y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \\
=& \frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\sqrt{b}\cdot u\omega)(x)(\sqrt{b(q_i)}-\sqrt{b(y)})g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \\
&+ \frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\sqrt{b}\cdot u\omega)(x)\sqrt{b(q_i)} \\
&\times (g(x-q_i)-g(x-y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \\
&+ \frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)}\omega(x)(\nabla\sqrt{b}(q_i)\cdot u(q_i) \\
&- \nabla\sqrt{b}(y)\cdot u(y))g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \\
&+ \frac{1}{N}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)}\omega(x)\nabla\sqrt{b}(q_i)\cdot u(q_i) \\
&\times (g(x-q_i)-g(x-y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x.
\end{aligned}$$

For the first integral, we use the Lipschitz regularity of $\sqrt{b}$ to bound

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\sqrt{b}\cdot u\omega)(x)(\sqrt{b(q_i)}-\sqrt{b(y)})g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \right| \\
&\qquad\qquad \leqslant C_b\eta \iint_{\mathbb{R}^2\times\mathbb{R}^2} |(\nabla\sqrt{b}\cdot u\omega)(x)g(x-y)|\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x.
\end{aligned}$$

Moreover for $y \in \partial B(q_i,\eta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(\nabla\sqrt{b}\cdot u\omega)(x)g(x-y)|\,\mathrm{d}x \\
&\leqslant \int_{B(y,1)} |(\nabla\sqrt{b}\cdot u\omega)(x)g(x-y)|\,\mathrm{d}x \\
&\quad + \int_{B(y,1)^c} |(\nabla\sqrt{b}\cdot u\omega)(x)g(x-y)|\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}&\leqslant \left\|\nabla\sqrt{b}\cdot u\omega\right\|_{L^\infty} \|g\|_{L^1(B(0,1))} + \int_{B(y,1)^c} |(\nabla\sqrt{b}\cdot u\omega)(x)|(|x|+|y|)\,\mathrm{d}x \\ &\leqslant C_b \|u\|_{L^\infty} \|\omega\|_{L^\infty} (1+|q_i|)\end{aligned}$$

since b satisfies Assumption 3.1.3. Therefore

$$\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2} (\nabla\sqrt{b}\cdot u\omega)(x)(\sqrt{b(q_i)}-\sqrt{b(y)})g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x\right| \\ \leqslant C_b \|u\|_{L^\infty} \|\omega\|_{L^\infty} (1+|q_i|)\eta.$$

The third integral can be bounded in the same way:

$$\begin{aligned}&\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)}\omega(x)(\nabla\sqrt{b}(q_i)\cdot u(q_i) - \nabla\sqrt{b}(y)\cdot u(y))g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x\right| \\ &\qquad\leqslant C_b \|u\|_{W^{1,\infty}}\,\eta \iint_{\mathbb{R}^2\times\mathbb{R}^2} |\sqrt{b(x)}\omega(x)g(x-y)|\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x \\ &\qquad\leqslant C_b \|u\|_{W^{1,\infty}} \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} (1+|q_i|)\eta.\end{aligned}$$

Summing over N we get that both the first and the third line can be bounded by

$$C_b \|u\|_{W^{1,\infty}} \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} (1+I(Q_N))\eta. \tag{3.6.7}$$

Now the second integral is equal to

$$\frac{1}{N}\sum_{i=1}^N \sqrt{b(q_i)} \int_{\mathbb{R}^2} (g*(\nabla\sqrt{b}\cdot u\omega)(q_i) - g*(\nabla\sqrt{b}\cdot u\omega)(y))\,\mathrm{d}\delta_{q_i}^{(\eta)}(y)$$

and thus by Morrey's inequality (see [14, Theorem 9.12]) its absolute value can be bounded by

$$C_{b,p}\eta^{1-\frac{2}{p}} \left\|\nabla g*(\nabla\sqrt{b}\cdot u\omega)\right\|_{L^p}$$

for any finite $p>2$. The fourth integral can be bounded in the same way by

$$C_{b,p}\eta^{1-\frac{2}{p}} \left\|\nabla g*(\sqrt{b}\omega)\right\|_{L^p}.$$

Using Hardy-Littlewood-Sobolev inequality (see for example [3, Theorem 1.7]) we have

$$C_b\eta^{1-\frac{2}{p}} \left\|\nabla g*(\sqrt{b}\omega)\right\|_{L^p} \leqslant C_b\eta^{1-\frac{2}{p}} \|\omega\|_{L^{\frac{2p}{p+2}}}. \tag{3.6.8}$$

Combining (3.6.7) and (3.6.8) we get that

$$|T_{2,1}| \leqslant C_b \|u\|_{W^{1,\infty}} \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} (1+I(Q_N))\eta^s$$

for some $0<s<1$. □

Now we bound $T_{2,2}$:

Claim 3.6.4.
$$|T_{2,2}| \leqslant C_b \|\nabla u\|_{L^\infty} \|\omega\|_{L^1 \cap L^\infty} \eta.$$

Proof of Claim 3.6.4. Let us recall that

$$\widetilde{\delta}_q^{(\eta)} = m_b(q,\eta) \frac{\mathrm{d}\delta_q^{(\eta)}}{\sqrt{b}}$$

and thus

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)b(y)}(u(x)-u(y))\cdot\nabla g(x-y)\omega(x)\,\mathrm{d}x\,\mathrm{d}(\delta_{q_i}-\widetilde{\delta}_{q_i}^{(\eta)})(y)\\
=&m_b(q_i,\eta)\iint_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{b(x)}(u(x)-u(y))\cdot\nabla g(x-y)\omega(x)\,\mathrm{d}x\,\mathrm{d}(\delta_{q_i}-\delta_{q_i}^{(\eta)})(y)\\
&+\left(1-\frac{m_b(q_i,\eta)}{\sqrt{b(q_i)}}\right)\int_{\mathbb{R}^2}\sqrt{b(x)b(q_i)}(u(x)-u(q_i))\cdot\nabla g(x-q_i)\omega(x)\,\mathrm{d}x.
\end{aligned}$$

The first integral is exactly the term defined in [71, Equation (4.10)] with $s=0$ and $m=0$ (remark that we can choose $m=0$ since no extension procedure is needed for $s=0$ and $d=2$, for more details we refer to the introduction of [71, Section 4]). It can be bounded by the right hand side of [71, Equation (4.24)] :

$$\begin{aligned}
C\|\nabla u\|_{L^\infty}\left\||\nabla|^{-1}(\sqrt{b}\omega)\right\|_{L^\infty}\eta &\leqslant C_b\|\nabla u\|_{L^\infty}\left\||\nabla|^{-1}\omega\right\|_{L^\infty}\eta\\
&\leqslant C_b\|\nabla u\|_{L^\infty}\|\omega\|_{L^1\cap L^\infty}\eta.
\end{aligned}$$

A proof of the last inequality can be found for example in [46, Lemma 1]. Now by (3.2.15) and the Lipschitz regularity of u we can bound the second line by

$$C_b\|\nabla u\|_{L^\infty}\|\omega\|_{L^1}\eta.$$

Combining the two upper equations we get

$$|T_{2,2}| \leqslant C_b \|\nabla u\|_{L^\infty} \|\omega\|_{L^1 \cap L^\infty} \eta.$$

□

Claim 3.6.5.

$$|T_{2,3}| \leqslant C_{b,s}\|u\|_{W^{1,\infty}}\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)}(1+I(Q_N))\eta^s.$$

Proof of Claim 3.6.5. We write $T_{2,3}$ as

$$T_{2,3} = \frac{1}{N}\sum_{i=1}^N\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2}\omega(x)u(x)\cdot(\nabla_x S_b(x,q_i)-\nabla_x S_b(x,y))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x\right.$$

$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} \omega(x)(u(q_i)-u(y))\cdot\nabla_x S_b(q_i,x)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x$$
$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} \omega(x)u(y)\cdot(\nabla_x S_b(q_i,x)-\nabla_x S_b(y,x))\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y)\,\mathrm{d}x\Bigg).$$

Using Claims (1) and (2) of Lemma 3.2.7, we get that for some $0<s<1$,

$$\begin{aligned}|T_{2,3}| \leqslant &\frac{1}{N}\sum_{i=1}^{N}\Bigg(C_{b,s}\,\|u\|_{L^\infty}\,\|\omega\|_{L^1}\,(1+|q_i|)\eta^s \\ &+\|\nabla u\|_{L^\infty}\,\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)}\,\eta \\ &+\|u\|_{L^\infty}\,\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)}\,\eta^s\Bigg) \\ \leqslant &C_{b,s}\,\|u\|_{W^{1,\infty}}\,\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)}\,(1+I(Q_N))\eta^s.\end{aligned}$$

□

Combining Claims 3.6.3, 3.6.4 and 3.6.5 we get that

$$|T_2| \leqslant C_{b,s}\,\|u\|_{W^{1,\infty}}\,\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\,(1+I(Q_N))\eta^s. \tag{3.6.9}$$

Now let us write T_3 as

$$\begin{aligned}T_3 =&\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(x)\cdot\nabla_x g_b(x,y) \\ &(\,\mathrm{d}\delta_{q_i}(x)\,\mathrm{d}\delta_{q_j}(y)-\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y)) \\ =&\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(x)\cdot\nabla\sqrt{b}(x)\sqrt{b(y)}g(x-y) \\ &(\,\mathrm{d}\delta_{q_i}(x)\,\mathrm{d}\delta_{q_j}(y)-\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y)) \\ &+\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\sqrt{b(x)b(y)}u(x)\cdot\nabla g(x-y) \\ &(\,\mathrm{d}\delta_{q_i}(x)\,\mathrm{d}\delta_{q_j}(y)-\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y)) \\ &+\frac{1}{N^2}\sum_{1\leqslant i,j\leqslant N}\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(x)\cdot\nabla_x S_b(x,y) \\ &(\,\mathrm{d}\delta_{q_i}(x)\,\mathrm{d}\delta_{q_j}(y)-\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y)) \\ =:&T_{3,1}+T_{3,2}+T_{3,3}.\end{aligned}$$

We bound the first term:

Claim 3.6.6.

$$|T_{3,1}| \leqslant C_b \|u\|_{L^\infty} |\mathcal{F}_b(Q_N,\omega)| + C_b \|u\|_{W^{1,\infty}} \left(\frac{g(\eta)}{N} \right.$$
$$\left. + I(Q_N)(\eta + N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} g(\eta)\eta \right).$$

Proof of Claim 3.6.6. We write

$$\begin{aligned}
T_{3,1} =& -\frac{1}{N^2}\sum_{i=1}^N \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla\sqrt{b}(x)\sqrt{b(y)}g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(y) \\
&+ \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} u(x)\cdot\nabla\sqrt{b}(x)\sqrt{b(y)}g(x-y) \\
&(\,\mathrm{d}\delta_{q_i}(x)\,\mathrm{d}\delta_{q_j}(y) - \,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y)) \\
=:& T_{3,1,1} + T_{3,1,2}.
\end{aligned}$$

By the definition of $\widetilde{\delta}_{q_i}^{(\eta)}$ (3.2.13) we have

$$\begin{aligned}
T_{3,1,1} =& -\frac{1}{N^2}\sum_{i=1}^N m_b(q_i,\eta)^2 \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\frac{\nabla\sqrt{b}(x)}{\sqrt{b(x)}}g(x-y)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)\,\mathrm{d}\delta_{q_i}^{(\eta)}(y) \\
=& -\frac{1}{N^2}\sum_{i=1}^N m_b(q_i,\eta)^2 \int_{\mathbb{R}^2} \frac{u(x)\cdot\nabla\sqrt{b}(x)}{\sqrt{b}(x)} g^{(\eta)}(x-q_i)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)
\end{aligned}$$

by Lemma 2.3.1. It follows by Assumption 3.1.3 that

$$|T_{3,1,1}| \leqslant \frac{C_b \|u\|_{L^\infty} g(\eta)}{N}. \tag{3.6.10}$$

Now we write

$$\begin{aligned}
T_{3,1,2} =& \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \Bigg((u\cdot\nabla\sqrt{b})(q_i)\sqrt{b(q_j)}g(q_i-q_j) \\
&- \iint_{\mathbb{R}^2\times\mathbb{R}^2} (u\cdot\nabla\sqrt{b})(x)\sqrt{b(y)}g(x-y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y) \Bigg) \\
=& \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} \Bigg((u\cdot\nabla\sqrt{b})(q_i)\sqrt{b(q_j)}g(q_i-q_j) \\
&- m_b(q_j,\eta)\int_{\mathbb{R}^2} (u\cdot\nabla\sqrt{b})(x) g^{(\eta)}(x-q_j)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x) \Bigg)
\end{aligned}$$

by the definition of $\widetilde{\delta}_{q_i}^{(\eta)}$ (3.2.13) and Lemma 2.3.1. Now,

$$T_{3,1,2} = \frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N} (u\cdot\nabla\sqrt{b})(q_i)\sqrt{b(q_j)}(g(q_i-q_j) - g^{(\eta)}(q_i-q_j))$$

$$
\begin{aligned}
&+\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(u\cdot\nabla\sqrt{b})(q_i)\sqrt{b(q_j)}\\
&\times\int_{\mathbb{R}^2}(g^{(\eta)}(q_i-q_j)-g^{(\eta)}(x-q_j))\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)\\
&+\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(u\cdot\nabla\sqrt{b})(q_i)\sqrt{b(q_j)}\int_{\mathbb{R}^2}g^{(\eta)}(x-q_j)\,\mathrm{d}(\delta_{q_i}^{(\eta)}-\widetilde{\delta}_{q_i}^{(\eta)})(x)\\
&+\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}\sqrt{b(q_j)}\\
&\times\int_{\mathbb{R}^2}((u\cdot\nabla\sqrt{b})(q_i)-(u\cdot\nabla\sqrt{b})(x))g^{(\eta)}(x-q_j)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\\
&+\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(\sqrt{b(q_j)}-m_b(q_j,\eta))\\
&\times\int_{\mathbb{R}^2}(u\cdot\nabla\sqrt{b})(x)g^{(\eta)}(x-q_j)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\\
&=:S_1+S_2+S_3+S_4+S_5.
\end{aligned}
$$

Since $g-g^{(\eta)}$ is nonnegative we can bound

$$
\begin{aligned}
|S_1|\leqslant&C_b\left\|u\right\|_{L^\infty}\frac{1}{N^2}\sum_{1\leqslant i\neq j\leqslant N}(g(q_i-q_j)-g^{(\eta)}(q_i-q_j))\\
\leqslant&C_b\left\|u\right\|_{L^\infty}|\mathcal{F}_b(Q_N,\omega)|+C_b\left\|u\right\|_{L^\infty}\Bigg(\frac{g(\eta)}{N}\\
&+I(Q_N)(\eta+N^{-1})+\left\|\omega\right\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}g(\eta)\eta\Bigg)
\end{aligned}
\tag{3.6.11}
$$

by Proposition 3.5.1. Now remark that if $|q_i-q_j|\geqslant 2\eta$ and $x\in\partial B(q_i,\eta)$,

$$
|q_j-x|\geqslant|q_i-q_j|-|q_i-x|\geqslant 2\eta-\eta\geqslant\eta
$$

and it follows by Lemma 2.3.1 that

$$
\begin{aligned}
\int_{\mathbb{R}^2}g^{(\eta)}(x-q_j)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)&=\int_{\mathbb{R}^2}g(x-q_j)\,\mathrm{d}\delta_{q_i}^{(\eta)}(x)\\
&=g^{(\eta)}(q_i-q_j).
\end{aligned}
$$

Hence we can write

$$
S_2=\frac{1}{N^2}\sum_{\substack{1\leqslant i\neq j\leqslant N\\|q_i-q_j|\leqslant 2\eta}}(u\cdot\nabla\sqrt{b})(q_i)\sqrt{b(q_j)}\int_{\mathbb{R}^2}(g^{(\eta)}(q_i-q_j)-g^{(\eta)}(x-q_j))\,\mathrm{d}\delta_{q_i}^{(\eta)}(x).
$$

Notice that if $|q_i-q_j|\leqslant 2\eta$ and $x\in\partial B(q_i,\eta)$, then

$$
|g^{(\eta)}(q_i-q_j)-g^{(\eta)}(x-q_j)|\leqslant\left\|\nabla g^{(\eta)}\right\|_{L^\infty}\eta=C\eta^{-1}\eta\leqslant C.
$$

Therefore,

$$
\begin{aligned}
|S_2| \leqslant & \frac{C_b \|u\|_{L^\infty}}{N^2} |\{(q_i, q_j); |q_i - q_j| \leqslant 2\eta\}| \\
\leqslant & C_b \|u\|_{L^\infty} |\mathcal{F}_b(Q_N, \omega)| + C_b \|u\|_{L^\infty} \Bigg(\frac{g(2\eta)}{N} \\
& + I(Q_N)(2\eta + N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} g(2\eta)2\eta \Bigg)
\end{aligned}
\tag{3.6.12}
$$

by Corollary 3.5.2 applied to $\varepsilon = 2\eta$.

By definition of $\widetilde{\delta}_{q_i}^{(\eta)}$ (3.2.13) we can write

$$
\begin{aligned}
S_3 = & \frac{1}{N^2} \sum_{1\leqslant i \neq j \leqslant N} (u \cdot \nabla \sqrt{b})(q_i) \sqrt{b(q_j)} \\
& \times \int_{\mathbb{R}^2} g^{(\eta)}(x - q_j) \left(1 - \frac{m_b(q_i, \eta)}{\sqrt{b(x)}} \right) \mathrm{d}\delta_{q_i}^{(\eta)}(x)
\end{aligned}
$$

and therefore

$$
|S_3| \leqslant \frac{C_b \|u\|_{L^\infty} g(\eta)}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^2} \left| \frac{m_b(q_i, \eta)}{\sqrt{b(x)}} - 1 \right| \mathrm{d}\delta_{q_i}^{(\eta)}(x).
$$

For $x \in \partial B(q_i, \eta)$, we have

$$
\begin{aligned}
\left| \frac{m_b(q_i, \eta)}{\sqrt{b(x)}} - 1 \right| & \leqslant C_b \left| m_b(q_i, \eta)^{-1} - \frac{1}{\sqrt{b(x)}} \right| \\
& \leqslant C_b \left| \int_{\mathbb{R}^2} \frac{\mathrm{d}\delta_{q_i}^{(\eta)}(y)}{\sqrt{b(y)}} - \frac{1}{\sqrt{b(x)}} \right| \\
& \leqslant C_b \eta
\end{aligned}
$$

since b is Lipschitz by Assumption 3.1.3. It follows that

$$
|S_3| \leqslant C_b \|u\|_{L^\infty} g(\eta)\eta. \tag{3.6.13}
$$

Now by regularity of u, b and (3.2.15), we get

$$
|S_4| + |S_5| \leqslant C_b \|u\|_{W^{1,\infty}} \eta g(\eta).
$$

Combining the upper inequality with (3.6.10), (3.6.11), (3.6.12) and (3.6.13) we obtain Claim 3.6.6. □

For the third term we have the following bound:

Claim 3.6.7. For s small enough, we have

$$
|T_{3,3}| \leqslant C_{b,s} \|u\|_{W^{1,\infty}} (1 + I(Q_N))\eta^s.
$$

Proof of Claim 3.6.7. We write

$$\begin{aligned}
T_{3,3} =& \frac{1}{N^2} \sum_{1\leqslant i,j\leqslant N} u(q_i)\cdot\nabla_x S_b(q_i,q_j) \\
& - \frac{1}{N^2} \sum_{1\leqslant i,j\leqslant N} \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\cdot\nabla_x S_b(x,y)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y) \\
=& \frac{1}{N^2} \sum_{1\leqslant i,j\leqslant N} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \Big(u(q_i)\cdot\nabla_x S_b(q_i,q_j) - u(q_i)\cdot\nabla_x S_b(q_i,y) \\
& + u(q_i)\cdot\nabla_x S_b(q_i,y) - u(q_i)\cdot\nabla_x S_b(x,y) \\
& + u(q_i)\cdot\nabla_x S_b(x,y) - u(x)\cdot\nabla_x S_b(x,y)\Big)\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|T_{3,3}| \leqslant& \frac{1}{N^2} \sum_{1\leqslant i,j\leqslant N} \|u\|_{L^\infty} |\nabla_x S_b(q_i,\cdot)|_{\mathcal{C}^{0,s}(B(q_j,1))}\eta^s \\
& + \frac{1}{N^2} \sum_{1\leqslant i,j\leqslant N} \|u\|_{L^\infty}\,\eta^s \int_{\mathbb{R}^2} |\nabla_x S_b(\cdot,y)|_{\mathcal{C}^{0,s}}\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y) \\
& + \frac{1}{N^2} \sum_{1\leqslant i,j\leqslant N} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \|u\|_{W^{1,\infty}}\,\eta|\nabla_x S_b(x,y)|\,\mathrm{d}\widetilde{\delta}_{q_i}^{(\eta)}(x)\,\mathrm{d}\widetilde{\delta}_{q_j}^{(\eta)}(y).
\end{aligned}$$

By Proposition 3.2.7, for s small enough we have

$$\begin{aligned}
|T_{3,3}| \leqslant& \frac{C_{b,s}}{N^2} \sum_{1\leqslant i,j\leqslant N} \|u\|_{L^\infty} |(1+|q_j|)\eta^s \\
& + \frac{C_{b,s}}{N^2} \sum_{1\leqslant i,j\leqslant N} \|u\|_{L^\infty}\,\eta^s(1+|q_j|) \\
& + \frac{C_b}{N^2} \sum_{1\leqslant i,j\leqslant N} \|u\|_{W^{1,\infty}}\,\eta(1+|q_j|) \\
\leqslant& C_{b,s}\,\|u\|_{W^{1,\infty}}\,(1+I(Q_N))\eta^s.
\end{aligned}$$

□

We are only remained to bound $T_{3,2}$:

Claim 3.6.8. For $\varepsilon > 2\eta$ small enough, we have

$$\begin{aligned}
|T_{3,2}| \leqslant& \frac{C_b}{N}\,\|\nabla u\|_{L^\infty} + \frac{C_b\eta\,\|\nabla u\|_{L^\infty}}{\varepsilon} + C_b\,\|\nabla u\|_{L^\infty}\Big(|\mathcal{F}_b(Q_N,\omega)| + \frac{g(\varepsilon)}{N} + \eta \\
& + I(Q_N)(\varepsilon+N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\,g(\varepsilon)\varepsilon\Big).
\end{aligned}$$

Proof of Claim 3.6.8. Let us denote

$$k_u(x,y) = (u(x) - u(y)) \cdot \nabla g(x-y)$$

and remark that

$$|k_u(x,y)| \leqslant C \left\| \nabla u \right\|_{L^\infty}. \tag{3.6.14}$$

Since ∇g is antisymmetric we can write $T_{3,2}$ as

$$T_{3,2} = \frac{1}{2N^2} \sum_{1 \leqslant i,j \leqslant N} \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta} \sqrt{b(x)b(y)} k_u(x,y) \, \mathrm{d}(\delta_{q_i} + \widetilde{\delta}_{q_i}^{(\eta)})(x) \, \mathrm{d}(\delta_{q_j} - \widetilde{\delta}_{q_j}^{(\eta)})(y).$$

Using the definition of $\widetilde{\delta}_{q_i}^{(\eta)}$ (3.2.13) we can write

$$\begin{aligned}
&\mathrm{d}(\delta_{q_i} + \widetilde{\delta}_{q_i}^{(\eta)})(x) \, \mathrm{d}(\delta_{q_j} - \widetilde{\delta}_{q_j}^{(\eta)})(y) \\
&\qquad = \mathrm{d}\left(\delta_{q_i} + \frac{m_b(q_i,\eta)}{\sqrt{b}} \delta_{q_i}^{(\eta)} \right)(x) \, \mathrm{d}\left(\delta_{q_j} - \frac{m_b(q_j,\eta)}{\sqrt{b}} \delta_{q_j}^{(\eta)} \right)(y) \\
&\qquad = \frac{m_b(q_i,\eta) m_b(q_j,\eta)}{\sqrt{b(x)b(y)}} \, \mathrm{d}(\delta_{q_i} + \delta_{q_i}^{(\eta)})(x) \, \mathrm{d}(\delta_{q_j} - \delta_{q_j}^{(\eta)})(y) \\
&\qquad\quad + \left(1 - \frac{m_b(q_i,\eta) m_b(q_j,\eta)}{\sqrt{b(q_i)b(q_j)}} \right) \mathrm{d}\delta_{q_i}(x) \, \mathrm{d}\delta_{q_j}(y) \\
&\qquad\quad + \frac{m_b(q_i,\eta)}{\sqrt{b(q_i)}} \left(1 - \frac{m_b(q_j,\eta)}{\sqrt{b(q_j)}} \right) \mathrm{d}\delta_{q_i}^{(\eta)}(x) \, \mathrm{d}\delta_{q_j}(y) \\
&\qquad\quad + \frac{m_b(q_j,\eta)}{\sqrt{b(y)}} \left(\frac{m_b(q_i,\eta)}{\sqrt{b(q_i)}} - 1 \right) \mathrm{d}\delta_{q_i}(x) \, \mathrm{d}\delta_{q_j}^{(\eta)}(y).
\end{aligned}$$

We will use some inequalities proved in [71] and Corollary 3.5.2 to control the first line, but let us begin by controling the three last remainders. Using the bound (3.6.14) and (3.2.15) we can bound

$$\begin{aligned}
T_{3,2,2} := &\frac{1}{2N^2} \sum_{1 \leqslant i,j \leqslant N} \iint_{(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta} \sqrt{b(x)b(y)} k_u(x,y) \\
&\left(\left(1 - \frac{m_b(q_i,\eta) m_b(q_j,\eta)}{\sqrt{b(q_i)b(q_j)}} \right) \mathrm{d}\delta_{q_i}(x) \, \mathrm{d}\delta_{q_j}(y) \right. \\
&+ \frac{m_b(q_i,\eta)}{\sqrt{b(q_i)}} \left(1 - \frac{m_b(q_j,\eta)}{\sqrt{b(q_j)}} \right) \mathrm{d}\delta_{q_i}^{(\eta)}(x) \, \mathrm{d}\delta_{q_j}(y) \\
&\left. + \frac{m_b(q_j,\eta)}{\sqrt{b(y)}} \left(\frac{m_b(q_i,\eta)}{\sqrt{b(q_i)}} - 1 \right) \mathrm{d}\delta_{q_i}(x) \, \mathrm{d}\delta_{q_j}^{(\eta)}(x) \right).
\end{aligned}$$

by

$$|T_{3,2,2}| \leqslant C_b \|\nabla u\|_{L^\infty} \eta. \tag{3.6.15}$$

We are remained to bound

$$\begin{aligned} T_{3,2,1} :=& \frac{1}{2N^2} \sum_{1 \leqslant i,j \leqslant N} m_b(q_i,\eta) m_b(q_j,\eta) \\ &\times \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} k_u(x,y)\, \mathrm{d}(\delta_{q_i} + \delta_{q_i}^{(\eta)})(x)\, \mathrm{d}(\delta_{q_j} - \delta_{q_j}^{(\eta)}(y). \end{aligned}$$

Using decomposition (4.26) and inequalities (4.27), (4.28) and (4.31) of [71] with $s=0$ and $m=0$ (remark that we can choose $m=0$ since no extension procedure is needed for $s=0$ and $d=2$, for more details we refer to the introduction of [71, Section 4]), we get that for any small parameter $\varepsilon > 2\eta$,

$$|T_{3,2,1}| \leqslant \frac{C_b}{N} \|\nabla u\|_{L^\infty} + \frac{C_b \|\nabla u\|_{L^\infty}}{N^2} |\{(q_i,q_j); |q_i - q_j| \leqslant \varepsilon\}| + \frac{C\eta \|\nabla u\|_{L^\infty}}{\varepsilon}.$$

Using Corollary 3.5.2, we get that

$$\begin{aligned} T_{3,2,1} \leqslant& \frac{C_b}{N} \|\nabla u\|_{L^\infty} + \frac{C_b \eta \|\nabla u\|_{L^\infty}}{\varepsilon} + C_b \|\nabla u\|_{L^\infty} \Big(|\mathcal{F}_b(Q_N,\omega)| \\ &+ \frac{g(\varepsilon)}{N} + I(Q_N)(\varepsilon + N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} g(\varepsilon)\varepsilon \Big). \end{aligned} \tag{3.6.16}$$

And we get Claim 3.6.8 by combining (3.6.16) with (3.6.15). □

We finish the proof of Proposition 3.6.1 using Decomposition (3.6.1), Inequalities (3.6.6), (3.6.9) and Claims 3.6.6, 3.6.7 and 3.6.8. That gives

$$\begin{aligned} \Big| \iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} & u(x)\cdot\nabla_x g_b(x,y)\, \mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i} - \omega\right)^{\otimes 2}(x,y) \Big| \\ \leqslant& C_b \|u\|_{W^{1,\infty}} \Big(|\mathcal{F}_b(Q_N,\omega)| + \frac{g(\eta)}{N} + I(Q_N)(\eta + N^{-1}) \\ &+ \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} g(\eta)\eta \Big) \\ &+ C_{b,s} \|u\|_{W^{1,\infty}} \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} (1 + I(Q_N))\eta^s \\ &+ C_b \|u\|_{L^\infty} \mathcal{F}_b(Q_N,\omega) + C_b \|u\|_{W^{1,\infty}} \Big(\frac{g(\eta)}{N} \\ &+ I(Q_N)(\eta + N^{-1}) + \|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty} g(\eta)\eta \Big) \\ &+ C_{b,s} \|u\|_{W^{1,\infty}} (1 + I(Q_N))\eta^s \\ &+ \frac{C_b}{N} \|\nabla u\|_{L^\infty} + \frac{C_b\eta \|\nabla u\|_{L^\infty}}{\varepsilon} + C_b \|\nabla u\|_{L^\infty} \Big(|\mathcal{F}_b(Q_N,\omega)| \end{aligned}$$

$$+\frac{g(\varepsilon)}{N}+\eta+I(Q_N)(\varepsilon+N^{-1})+\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\,g(\varepsilon)\varepsilon\Bigg).$$

Choosing $\varepsilon = N^{-1}$ and $\eta = N^{-2}$, and since $\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}$ is bounded by below (because ω is a probability density) we get that

$$\begin{aligned}\Bigg|\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta} & u(x)\cdot\nabla_x g_b(x,y)\,\mathrm{d}\left(\frac{1}{N}\sum_{i=1}^N \delta_{q_i}-\omega\right)^{\otimes 2}(x,y)\Bigg|\\ &\leqslant C_b\,\|u\|_{W^{1,\infty}}\,|\mathcal{F}_b(Q_N,\omega)|\\ &\quad + C_b(1+\|u\|_{W^{1,\infty}})\,\|\omega\|_{L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty}\,(1+I(Q_N))N^{-\beta}\end{aligned}$$

for some $0<\beta<1$. □

3.7 Mean-field limit

In this section we prove the mean-field limit Theorem 3.1.6. For this purpose let us first prove the following estimates:

Theorem 3.7.1. *If ω is a weak solution of (3.1.2) with initial datum ω_0 (in the sense of Definition 3.1.1) that satisfies Assumption 3.1.4 and if $I_N(0)$ is bounded, there exists a constant*

$$A := A\left(b, T, \|u\|_{L^\infty([0,T],W^{1,\infty})}, \|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}, \sup_N I_N(0)\right)$$

such that for every $t\in[0,T]$,

$$|\mathcal{F}_{b,N}(t)| \leqslant A(|\mathcal{F}_{b,N}(0)| + (1+|E_N(0)|)(N^{-\beta}+|\alpha_N-\alpha|)). \tag{3.7.1}$$

If $\overline{\omega}$ is a weak solution of (3.1.5) with initial datum ω_0 (in the sense of Definition 3.1.2) that satisfies Assumption 3.1.4 and if $\overline{I_N}(0)$ is bounded, there exists a constant

$$\begin{aligned}B := B\Bigg(& b, T, \|\overline{\omega}\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)},\\ & \|\nabla g * \overline{\omega}\|_{L^\infty([0,T],W^{1,\infty})}, \sup_N \overline{I_N}(0)\Bigg)\end{aligned}$$

such that for every $t\in[0,T]$,

$$|\overline{\mathcal{F}}_{b,N}(t)| \leqslant B(|\overline{\mathcal{F}}_{b,N}(0)| + (1+|\overline{E_N}(0)|)(N^{-\beta}+\alpha_N^{-1})). \tag{3.7.2}$$

Proof. By Proposition 3.4.1, we have that for almost every $t\in[0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{b,N}(t)$$

$$
\begin{aligned}
=&2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\left(u(t,x)-\alpha\frac{\nabla^\perp b(x)}{b(x)}\right)\cdot\nabla_x g_b(x,y)\,\mathrm{d}(\omega(t)-\omega_N(t))^{\otimes 2}(x,y)\\
&+2(\alpha_N-\alpha)\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\frac{\nabla^\perp b(x)}{b(x)}\cdot\nabla_x g_b(x,y)\,\mathrm{d}\omega_N(t,x)\,\mathrm{d}(\omega(t)-\omega_N(t))(y)\\
=:&L_1+2(\alpha_N-\alpha)L_2.
\end{aligned}
$$

Using Proposition 3.6.1, we have

$$
\begin{aligned}
|L_1|\leqslant &C_b\left\|u-\alpha\frac{\nabla^\perp b}{b}\right\|_{L^\infty([0,T],W^{1,\infty})}|\mathcal{F}_b(Q_N,\omega)|\\
&+C_b\left(1+\left\|u-\alpha\frac{\nabla^\perp b}{b}\right\|_{L^\infty([0,T],W^{1,\infty})}\right)\\
&\times\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}(1+I_N(t))N^{-\beta}.
\end{aligned}
$$

By Proposition 3.3.1, we have

$$
I_N(t)\leqslant C_{b,T}(1+|E_N(0)|+I_N(0))
$$

since (α_N) is bounded (here we consider the case $\alpha_N\underset{N\to+\infty}{\longrightarrow}\alpha$). Therefore,

(3.7.3)

$$
\begin{aligned}
|L_1|\leqslant &C_b\big(1+\|u\|_{L^\infty([0,T],W^{1,\infty})}\big)|\mathcal{F}_b(Q_N,\omega)|+C_{b,T}\left(1+\|u\|_{L^\infty([0,T],W^{1,\infty})}\right)\\
&\times\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}(1+I_N(0)+|E_N(0)|)N^{-\beta}.
\end{aligned}
$$

Now

$$
\begin{aligned}
L_2=&\frac{1}{N}\sum_{i=1}^N\frac{\nabla^\perp b(q_i)}{b(q_i)}\\
&\cdot\left[\int_{\mathbb{R}^2\setminus\{q_i\}}\sqrt{b(q_i)b(y)}\nabla g(q_i-y)\,\mathrm{d}\Big(\omega(t)-\frac{1}{N}\sum_{j=1}^N\delta_{q_j(t)}\Big)\right.\\
&\left.+\int_{\mathbb{R}^2\setminus\{q_i\}}\nabla_x S_b(q_i,y)\,\mathrm{d}\Big(\omega(t)-\frac{1}{N}\sum_{j=1}^N\delta_{q_j(t)}\Big)\right]\\
=:&L_{2,1}+L_{2,2}+L_{2,3}
\end{aligned}
$$

with

(3.7.4)
$$
\begin{aligned}
|L_{2,1}|&=\left|\frac{1}{N}\sum_{i=1}^N\frac{\nabla^\perp b(q_i)}{\sqrt{b(q_i)}}\cdot\int_{\mathbb{R}^2}\nabla g(q_i-y)\sqrt{b(y)}\omega(t,y)\,\mathrm{d}y\right|\\
&\leqslant C_b\|\omega\|_{L^\infty([0,T],L^1\cap L^\infty)}
\end{aligned}
$$

(for the last inequality see for example [46, Lemma 1]). For the second term

$$L_{2,2} = -\frac{1}{N^2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sqrt{b(q_i)b(q_i)}\frac{\nabla^{\perp}b(q_i)}{b(q_i)}\cdot\nabla g(q_i-q_j).$$

We can bound it as in (3.3.7) to get

$$|L_{2,2}| \leqslant C_b. \tag{3.7.5}$$

For the last term, we use Claim (1) of Lemma 3.2.7 to get

$$\begin{aligned}
|L_{2,3}| &= \left|\frac{1}{N}\sum_{i=1}^{N}\frac{\nabla^{\perp}b(q_i)}{b(q_i)}\cdot\int_{\mathbb{R}^2}\nabla_x S_b(q_i,y)\,\mathrm{d}\Big(\omega(t)-\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\delta_{q_j(t)}\Big)(y)\right|\\
&\leqslant C_b\int_{\mathbb{R}^2}(1+|y|)\,\mathrm{d}\Big(\omega(t)+\frac{1}{N}\sum_{\substack{j=1\\j\neq i}}^{N}\delta_{q_j(t)}\Big)(y)\\
&\leqslant C_b(\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x))}+I_N(t))\\
&\leqslant C_{b,T}(\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x))}+1+I_N(0)+|E_N(0)|)
\end{aligned}$$

by Proposition 3.3.1. Combining the upper inequality with (3.7.3), (3.7.4) and (3.7.5) we get that for almost every $t\in[0,T]$,

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{b,N}(t)\right|\\
\leqslant & C_b(1+\|u\|_{L^\infty([0,T],W^{1,\infty})})|\mathcal{F}_{b,N}(t)|+C_b\left(1+\|u\|_{L^\infty([0,T],W^{1,\infty})}\right)\\
&\times\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}(1+I_N(0)+|E_N(0)|)N^{-\beta}\\
&+C_{b,T}|\alpha_N-\alpha|\Big(\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}+1+I_N(0)+|E_N(0)|\Big).
\end{aligned}$$

Therefore there exists a constant A depending only on the quantities b, T, $\|u\|_{L^\infty([0,T],W^{1,\infty})}$, $\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}$ and $I_N(0)$ (which is uniformly bounded in N by assumption) such that for almost every $t\in[0,T]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{b,N}(t)\right|\leqslant A(|\mathcal{F}_{b,N}(t)|+(1+|E_N(0)|)(N^{-\beta}+|\alpha_N-\alpha|)).$$

By Grönwall's lemma (up to taking another constant A depending on the same quantities) we get (3.7.1).

Now let us study the rescaled regime where $\alpha_N \underset{N\to+\infty}{\longrightarrow} +\infty$. By Proposition 3.4.2 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{F}}_{b,N}(t) = -2\iint_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta}\frac{\nabla^{\perp}b(x)}{b(x)}\cdot\nabla_x g_b(x,y)\,\mathrm{d}(\overline{\omega}(t)-\overline{\omega}_N(t))^{\otimes 2}(x,y)$$

$$+\frac{2}{N^2\alpha_N}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{v(t,\overline{q_i})}{b(\overline{q_i})}\cdot\nabla_x g_b(\overline{q_i},\overline{q_j})$$
$$=:L_1+L_2.$$

The first term can be bounded by Proposition 3.6.1:

$$\begin{aligned}|L_1|\leqslant & C_b\left\|\frac{\nabla b}{b}\right\|_{W^{1,\infty}}|\overline{\mathcal{F}}_{b,N}(t)|+C_b\left(1+\left\|\frac{\nabla b}{b}\right\|_{W^{1,\infty}}\right)\\ &\times\|\overline{\omega}\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}(1+I(\overline{Q_N}))N^{-\beta}\\ \leqslant & C_b|\overline{\mathcal{F}}_{b,N}(t)|+C_{b,T}\|\overline{\omega}\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}\\ &\times(1+\overline{I_N}(0)+|\overline{E_N}(0)|)N^{-\beta}\end{aligned}\tag{3.7.6}$$

where we used Proposition 3.3.1 in the last inequality. We split the second line in three terms:

$$\begin{aligned}L_2=&\frac{2}{N^2\alpha_N}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{v(t,\overline{q_i})}{b(\overline{q_i})}\cdot\frac{\nabla b(\overline{q_i})}{2\sqrt{b(\overline{q_i})}}\sqrt{b(\overline{q_j})}g(\overline{q_i}-\overline{q_j})\\ &+\frac{2}{N^2\alpha_N}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{v(t,\overline{q_i})}{b(\overline{q_i})}\cdot\nabla g(\overline{q_i}-\overline{q_j})\sqrt{b(\overline{q_i})b(\overline{q_i})}\\ &+\frac{2}{N^2\alpha_N}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{v(t,\overline{q_i})}{b(\overline{q_i})}\cdot\nabla_x S_b(\overline{q_i},\overline{q_j})\\ =:&L_{2,1}+L_{2,2}+L_{2,3}.\end{aligned}$$

We can bound the first term by

$$|L_{2,1}|\leqslant\frac{C_b}{N^2\alpha_N}\|v\|_{L^\infty}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}|g(\overline{q_i}-\overline{q_j})|$$

and applying Lemma 3.2.3 we get

$$\|v\|_{L^\infty}=\|\nabla G_b[\overline{\omega}]\|_{L^\infty}\leqslant C_b\|\overline{\omega}\|_{L^\infty([0,T],L^1\cap L^\infty)}.$$

We can bound $\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}|g(\overline{q_i}-\overline{q_j})|$ as we did for Inequality (3.3.10) to get

$$|L_{2,1}|\leqslant C_b\|\overline{\omega}\|_{L^\infty([0,T],L^1\cap L^\infty)}(1+|\overline{E_N}|+\overline{I_N})\alpha_N^{-1}.$$

The second term $L_{2,2}$ can be bounded as in (3.3.7) to get

$$|L_{2,2}| \leqslant C_b(1 + \|v\|_{L^\infty([0,T],W^{1,\infty})})\alpha_N^{-1}$$

and the last term can be bounded directly using Claim (1) of Lemma 3.2.7:

$$|L_{2,3}| \leqslant C_b \|v\|_{L^\infty} (1 + \overline{I_N})\alpha_N^{-1} \leqslant C_b \|\overline{\omega}\|_{L^\infty([0,T],L^1\cap L^\infty)} (1 + \overline{I_N})\alpha_N^{-1}.$$

Combining these three inequalities with (3.7.6) and using Proposition 3.3.1 to bound $\overline{I_N}$ we get that for almost every $t \in [0,T]$,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{F}}_{b,N}(t)\right| \leqslant & C_b|\overline{\mathcal{F}}_{b,N}(t)| + C_b(\|\overline{\omega}\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)} \\ & + \|v\|_{L^\infty([0,T],W^{1,\infty})}) \\ & \times (1 + |\overline{I_N}(0)| + |\overline{E_N}(0)|)(N^{-\beta} + \alpha_N^{-1}).\end{aligned}$$

And therefore there exists a constant B depending only on the quantities b, T,$\|\omega\|_{L^\infty([0,T],L^1((1+|x|)\,\mathrm{d}x)\cap L^\infty)}$ and $\overline{I_N}(0)$ (which is uniformly bounded in N by assumption) such that for almost every $t \in [0,T]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{F}}_{b,N}(t)\right| \leqslant B(|\overline{\mathcal{F}}_{b,N}(t)| + (1 + |\overline{E_N}(0)|)(N^{-\beta} + \alpha_N^{-1})).$$

By Grönwall's lemma (up to taking another constant B depending on the same quantities) we get (3.7.2). □

Proof of Theorem 3.1.6. By Corollary 3.5.4, weak-$*$ convergence and convergence of the interaction energy gives that $(\mathcal{F}_{b,N}(0))$ and $(\overline{\mathcal{F}_{b,N}}(0))$ converge to zero. Using convergence of the interaction energy we also get that $|E_N(0)|$ and $|\overline{E_N}(0)|$ are bounded. Thus by Inequalities (3.7.1) and (3.7.2) we get that for any $t \in [0,T]$ $(\mathcal{F}_{b,N}(t))$ and $(\overline{\mathcal{F}_{b,N}}(t))$ converge to zero and the theorem follows by Corollary 3.5.4. □

Notations

Constantes

— C désigne une constante générique dont la valeur peut changer d'une ligne d'une inégalité à la suivante. On notera $C(A, B)$ ou $C_{A,B}$ une constante dépendant de paramètres A et B.
— Pour $1 \leqslant p \leqslant +\infty$, p' désigne l'exposant dual de p (c'est à dire tel que $\frac{1}{p} + \frac{1}{p'} = 1$).

Opérations sur les vecteurs

Pour des vecteurs $x, y \in \mathbb{R}^2$ et des matrices $A, B \in \mathcal{M}_2(\mathbb{R})$, on note :

— $x \cdot y$ le produit scalaire entre x et y et $|x|$ la norme euclidienne de x,
— $A : B := \sum_{1 \leqslant i,j \leqslant 2} A_{i,j} B_{i,j}$ le produit scalaire entre A et B,
— $x^\perp := (-x_2, x_1)$,
— $<x> := (1 + |x|^2)^{\frac{1}{2}}$.
— Pour $X_N = (x_1, ..., x_N) \in (\mathbb{R}^2)^N$, on note $I(X_N) := \frac{1}{N} \sum_{i=1}^{N} |x_i|^2$.

Espaces de fonctions et normes

Considérons des espace de Banach A, B et un espace métrique E.

— B^* désigne le dual de B.
— Si $F(E, \mathbb{R})$ est un certain espace de fonctions de E dans $\mathbb{R}$, on notera $F(E) := F(E, \mathbb{R})$.
— Pour $k \in \mathbb{N}$, $\mathcal{C}^k$ désigne l'espace des fonctions de classe $\mathcal{C}^k$.
— $\mathcal{C}^{0,\theta}(\mathbb{R}^2)$ désigne l'espace des fonctions θ-Hölder à valeurs réelles, muni de la norme

$$\|f\|_{\mathcal{C}^{0,\theta}} := \|f\|_{L^\infty} + |f|_{\mathcal{C}^{0,\theta}}$$

avec

$$|f|_{\mathcal{C}^{0,\theta}} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

— Pour $k \in \mathbb{R}$, H^k désigne l'espace de Sobolev $W^{k,2}$,
— $\dot{H}^{-1}(\mathbb{R}^2)$ désigne l'ensemble des distributions tempérées f telles que $\widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ et telles que

$$\|f\|^2_{\dot{H}^{-1}(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \frac{|\widehat{f}(\xi)|^2}{|\xi|^2} < +\infty.$$

— $H^s_{\mathrm{ul}}(\mathbb{R}^2)$ désigne l'espace des fonctions localement uniformément H^s, c'est à dire les fonctions f telles que

$$\|f\|_{H^s_{\mathrm{ul}}(\mathbb{R}^2)} := \sup_{x\in\mathbb{R}^2} \|f\|_{H^s(B(x,1))} < +\infty.$$

— Pour $1 < p < \infty$, $W^{2,p}_{-1}(\mathbb{R}^2)$ désigne l'espace de Sobolev à poids défini par

$$W^{2,p}_{-1}(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2)\ ;\ \forall \alpha \in \mathbb{N}^2, |\alpha| \leqslant 2, <\cdot>^{|\alpha|-1} D^\alpha u \in L^p(\mathbb{R}^2)\}$$

et muni de la norme

$$\|u\|_{W^{2,p}_{-1}} := \left(\sum_{|\alpha|\leqslant 2} \left\| <\cdot>^{|\alpha|-1} D^\alpha u \right\|^p_{L^p} \right)^{\frac{1}{p}}.$$

— Pour $1 \leqslant p \leqslant \infty$, on notera

$$L^p_T B := L^p([0,T], B).$$

On utilisera la même convention pour les espaces $\mathcal{C}^k_T B$ ou $W^{k,p}_T B$.
— Si $B = A^*$, on notera $\mathcal{C}^0([0,T], B{-}w^*)$ l'espace des fonctions continues de $[0,T]$ dans B muni de la topologie faible-$*$.

Opérateurs différentiels

Considérons $f : \mathbb{R}^d \to \mathbb{R}$, u et v sont deux champs de vecteurs sur $\mathbb{R}^d$. On note :

— ∇ l'opérateur gradient : $\nabla f = (\partial_1 f, ..., \partial_d f)$,
— $(u\cdot\nabla)v$ le champ de la vecteur dont la i-ème coordonnée est $u \cdot \nabla v_i$,
— $\mathrm{div}(u) = \displaystyle\sum_{i=1}^d \partial_i u_i$ la divergence de u,

— $\Delta f = \sum_{i=1}^{d} \partial_{ii} f$ le laplacien de f.
— Pour $d = 2$, $\mathrm{curl}(u) = \partial_1 u_2 - \partial_2 u_1$ le rotationnel de u,
— Pour $d = 2$, $[f, f]$ désigne le tenseur défini par

$$[f, f]_{i,j} = 2\partial_i f \partial_j f - |\nabla f|^2 \delta_{i,j}.$$

Mesures et espaces de mesures

On considère $x \in \mathbb{R}^2$ et $0 < \eta < 1$.

— $\mathcal{P}(\mathbb{R}^2)$ désigne l'espace des mesures de probabilités sur $\mathbb{R}^2$.
— Si ν est une mesure, on note

$$\nu^{\otimes 2} := \nu \otimes \nu.$$

— δ_x est la masse de dirac en x.
— $\delta_x^{(\eta)}$ est la mesure de probabilité uniforme sur le cercle $\partial B(x, \eta)$.
— Dans le Chapitre 3, on note

$$\widetilde{\delta}_x^{(\eta)} := m_b(x, \eta) \frac{\mathrm{d}\delta_x^{(\eta)}}{\sqrt{b}}$$

où b est la fonction de profondeur satisfaisant l'hypothèse 3.1.3 et

$$m_b(y, \eta) := \left(\int \frac{\mathrm{d}\delta_y^{(\eta)}}{\sqrt{b}} \right)^{-1}.$$

Fonctions et ensembles

On note :

— $\mathbf{1}_\Omega$ la fonction indicatrice d'un ensemble Ω,
— Δ la diagonale de $(\mathbb{R}^2)^2$:

$$\Delta := \{(x, x) \; ; \; x \in \mathbb{R}^2\},$$

— $\mathrm{supp}(f)$ le support d'une fonction f,
— g l'opposé du noyau du laplacien sur le plan :

$$g(x) := -\frac{1}{2\pi} \ln |x|,$$

— $g^{(\eta)}$ la fonction définie par

$$g^{(\eta)}(x) = \begin{cases} -\dfrac{1}{2\pi} \ln(\eta) & \text{si } |x| \leqslant \eta \\ g(x) & \text{si } |x| \geqslant \eta, \end{cases}$$

- b la fonction de topographie considérée dans le Chapitre 3 qui satisfait 3.1.3,
- S_b la solution du problème elliptique (3.2.9),
- $g_b(x,y) = \sqrt{b(x)b(y)}g(x-y) + S_b(x,y)$ est le noyau du problème elliptique (3.2.2) construit en Section 3.2.

Fonctionnelles

- Si ρ est une fonction continue sur $\mathbb{R}^2$ à support compact on notera

  $$R[\rho] := \sup\left\{|x|\ ;\ x \in \mathbb{R}^2, \rho(x) \neq 0\right\}$$

  et

  $$R_T[\rho] := \sup_{0 \leqslant t \leqslant T} R[\rho(t)]$$

  si ρ dépends du temps.
- $\mathcal{F}$ désigne la fonctionnelle d'énergie modulée définie en (1.2.8).
- $\mathcal{F}_b$ désigne la fonctionnelle d'énergie modulée des lacs définie en (1.3.2).
- $\mathcal{H}$ désigne la fonctionnelle d'énergie modulée du modèle de spray définie en (1.3.1).
- G_b désigne l'opérateur qui a ω associe la solution ψ du problème elliptique $-\operatorname{div}\left(\frac{1}{b}\nabla\psi\right) = \omega$ (voir la Proposition 3.2.8).
- Pour $0 < \eta < 1$, on note

  $$H_{N,\eta} := G_b\left[\frac{1}{N}\sum_{i=1}^{N}\widetilde{\delta}_{q_i}^{(\eta)} - \omega\right].$$

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